CONFORMAL AND QUASICONFORMAL CATEGORICAL REPRESENTATION OF HYPERBOLIC RIEMANN SURFACES

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Abstract. In this paper, we consider various categories of hyperbolic Riemann surfaces and show, in various cases, that the *conformal* or *quasiconformal* structure of the Riemann surface may be reconstructed, up to possible confusion between holomorphic and anti-holomorphic structures, in a natural way from such a category. The theory exposed in the present paper is motivated partly by a classical result concerning the categorical representation of sober topological spaces, partly by previous work of the author concerning the categorical representation of arithmetic log schemes, and partly by a certain *analogy with p-adic anabelian geometry* — an analogy which the theory of the present paper serves to render more explicit.

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Introduction

In this paper, we continue our study [cf., [Mzk2], [Mzk10]] of the topic of *representing* various objects that appear in conventional *arithmetic geometry* by means of *categories*. As discussed in [Mzk2], [Mzk10], this point of view is partially motivated by the *anabelian philosophy* of Grothendieck [cf., e.g., [Mzk3], [Mzk4], [Mzk5]], and, in particular, by the more recent work of the author on *absolute anabelian geometry* [cf. [Mzk6], [Mzk7], [Mzk8], [Mzk9], [Mzk11], [Mzk12]].

One way to think about anabelian geometry is that it concerns the issue of representing schemes by means of categories [i.e., Galois categories] that capture certain aspects of the [étale] *topology* of the scheme [i.e., its fundamental group]. From this point of view, another important, albeit elementary, example of the issue

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of representing a "space" by means of a *"category of topological origin"* is the well-known example of the *category of open subsets of a sober topological space* [cf., e.g., [Mzk2], Theorem 1.4; [Mzk10], Proposition 4.1]. In some sense, this example is the example that motivated the construction of the categories appearing in the present paper.

The *main results* of this paper may be summarized as follows:

(1) The holomorphic structure of a hyperbolic Riemann surface of finite type may be reconstructed, up to possible confusion with the corresponding anti-holomorphic structure, from a certain *category of localizations* of the Riemann surface that includes the *upper half-plane uniformization* of the Riemann surface, together with its natural $PSL_2(\mathbb{R})$*-action* [cf. Theorem 1.12].

(2) Given a hyperbolic Riemann surface of finite type equipped with a nonzero *logarithmic square differential*, one may define certain *categories of parallelograms, rectangles, or squares* associated to this data. Then [isomorphism classes of] equivalences between corresponding categories of parallelograms (respectively, rectangles; squares) are in natural bijective correspondence with *[quasiconformal] Teichmüller mappings* (respectively, *conformal mappings*) between such Riemann surfaces equipped with differentials, again up to possible confusion between holomorphic and anti-holomorphic structures [cf. Theorem 2.3].

Here, we note that the categories of (2) are especially close to the "categories of open subsets of a sober topological space" referred to above — i.e., roughly speaking, instead of considering *all* the open subsets of the Riemann surface, one restricts oneself to those which are *"parallelograms"* (or, alternatively, *"rectangles"*, or *"squares"*), in a sense determined by the *natural parameters* [i.e., of Teichmüller theory — cf., e.g., [Lehto], Chapter IV, §6.1] associated to the given square differential.

On the other hand, from another point of view, the main motivation for the results obtained in this paper came from the analogy with *p-adic anabelian geometry.* This analogy has been pointed out previously by the author [cf., e.g., [Mzk1], Introduction, §0.10; [Mzk5], §3]. In some sense, however, the theory of the present paper allows one to make this analogy *more explicit.* Indeed, at the level of *"objects under consideration"* the theory of the present paper suggests a certain *"dictionary"*, as summarized in Table 1 below.

The first two non-italicized rows of Table 1 are motivated by the fact that the datum of a nonzero *logarithmic square differential* may be thought of, in the context of Teichmüller theory, as the datum of a *geodesic* in Teichmüller space. In particular, if one thinks of oneself as only knowing the differential *up to a nonzero complex multiple* [cf. Theorem 2.3], then one is, in essence, working with a *"complex Teichmüller geodesic"*. Moreover, just as such a "complex geodesic" is of "holomorphic dimension" one and "real/topological dimension" two, the spectrum of the ring of integers of a p-adic local field K is of algebraic dimension one, while the absolute Galois group G_K of the p-adic local field K is of cohomological dimension two. This observation also motivates the point of view of the third non-italicized row of Table 1, which is also discussed in [Mzk1], Introduction, §0.10. From the point of view of this third non-italicized row of Table 1, the conformal structure may be thought of as the *metric, or "angular", structure* of the $\mathbb{S}^1$ acting by rotations locally on the surface. On the other hand, from the point of view of p-adic anabelian geometry, one may completely recover the algebraic structure of the p-adic curve in question [cf. the main result of [Mzk4]], so long as one restricts oneself to working with *geometric* isomorphisms [i.e., isomorphisms arising from isomorphisms of fields] of the absolute Galois groups of the p-adic local fields in question. Moreover, as one sees from the theory of [Mzk3], this geometricity condition corresponds to the *preservation of the metric structure* of the copy of the units $\mathcal{O}_K^\times$ inside the abelianization G_K^{ab} of G_K [more precisely, the preservation of such metric structures for all open subgroups of G_K].

complex case	*p-adic case*
the given Riemann surface	the logarithmic special fiber
a complex Teichmüller geodesic originating from the given Riemann surface	a lifting of the special fiber to a hyperbolic curve over a p-adic local field K
action of $\mathbb{C}^\times$ on the surface by rotations ($\mathbb{S}^1 \subseteq \mathbb{C}^\times$) and flows ($\mathbb{R}^\times \subseteq \mathbb{C}^\times$)	action of the absolute Galois group G_K on [the Galois category associated to] the profinite geometric fundamental group
squares, rectangles, as opposed to parallelograms — i.e., preservation of the metric structure of $\mathbb{S}^1$	preservation of the metric structure of the copy of $\mathcal{O}_K^\times$ in G_K^{ab}

Table 1: Dictionary of objects under consideration

This "dictionary of objects under consideration" then suggests a *"dictionary of results"*, as summarized in Table 2 below. The analogy between the *"p-adic Teichmüller theory"* of [Mzk1] [and, in particular, the canonical representation constructed in this theory] and the upper half-plane uniformization of a hyperbolic Riemann surface of finite type is one of the *cornerstones* of the theory of [Mzk1]; in particular, a lengthy discussion of this analogy may be found in the Introduction to [Mzk1]. Also, relative to the issue of "reconstructing the original hyperbolic curve or Riemann surface", it is interesting to note that just as Theorem 1.12 does not require the datum of a logarithmic square differential, the absoluteness of canonical liftings [cf. [Mzk7], Theorem 3.6] only involves the datum of the logarithmic special fiber — i.e., there is no "choice" of a p-adic lifting involved [just as there is no "choice" of a complex Teichmüller geodesic in Theorem 1.12]. By contrast, just as the results on the left-hand side of the second and third non-italicized rows of Table 2 *do* involve the choice of such a complex Teichmüller geodesic, the hyperbolic curves involved on the right-hand side of the second and third non-italicized rows of Table 2 require the *choice* of a p-adic lifting of the logarithmic special fiber. As suggested by the dictionary of Table 1, the *"preservation of the metric structure of the units"* [i.e., $\mathbb{S}^1 \subseteq \mathbb{C}^\times$ or $\mathcal{O}_K^\times \subseteq G_K^{\mathrm{ab}}$] corresponds to complete reconstruction of the conformal structure of the Riemann surface or the algebraic structure of the p-adic curve in the second and third non-italicized rows of Table 2. On the other hand, reconstruction of the quasiconformal structure of the Riemann surface [essentially a *topological* invariant] corresponds, in the final row of Table 2, to the reconstruction of the dual semi-graph [also essentially a *topological* invariant] of the logarithmic special fiber, in the *absence* of the "preservation of the metric structure of the units". Also, it is interesting to note that the theory of the first non-italicized row of Table 2 is *not* functorial with respect to ramified coverings of the Riemann surface/non-admissible coverings of the p-adic hyperbolic curve, whereas the theory of the latter three non-italicized rows of Table 2 *is* functorial with respect to such coverings.

complex case	*p-adic case*
categorical representation via the upper half-plane uniformization [cf. Theorem 1.12]	the canonical representation of p-adic Teichmüller theory, the absoluteness of canonical liftings [cf. [Mzk1]; [Mzk7], Theorem 3.6]
conformal structure via categories of rectangles [cf. Theorem 2.3, (iii)]	relative p-adic profinite version of the Grothendieck Conjecture [cf. [Mzk4], Theorem A]
conformal structure via categories of squares [cf. Theorem 2.3, (iii)]	relative p-adic pro-p version of the Grothendieck Conjecture [cf. [Mzk4], Theorem A]
quasiconformal structure via categories of parallelograms [cf. Theorem 2.3, (ii)]	reconstruction of dual semi-graph of logarithmic special fiber via absolute p-adic pro-prime-to-p anabelian geometry or its tempered analogue [cf. [Mzk6], Lemma 2.3; [Mzk11], Corollary 3.11]

Table 2: Dictionary of results

Here, we remark that although it is quite possible that the *relative* p-adic profinite [or pro-p] versions of the Grothendieck Conjecture proven in [Mzk4] admit *absolute* generalizations [cf., e.g., [Mzk12], Corollary 2.12], if [as on the right-hand side of the fourth non-italicized row of Table 2] one restricts oneself to the *pro-prime-to-p* portion of the geometric fundamental group, then there is *no hope* [cf. the unbridgeable gap between conformal and quasiconformal structures!] of recovering the generic fiber of the original p-adic curve from the outer Galois action on the pro-prime-to-p geometric fundamental group, since this outer Galois action is completely *determined by the logarithmic special fiber*.

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Section 0: Notations and Conventions

Numbers:

The notation $\mathbb{Z}$ (respectively, $\mathbb{R}$; $\mathbb{C}$) will be used to denote the set of *rational integers* (respectively, *real numbers*; *complex numbers*).

Topological Groups:

A homomorphism of topological groups $G \to H$ will be called *dense* if the image of G is dense in H.

A topological group G will be called *tempered* [cf. [Mzk11], Definition 3.1, (i)] if G is isomorphic, as a topological group, to an inverse limit of an inverse system of surjections of countable discrete topological groups.

Categories:

Let $\mathcal{C}$ be a *category*. We shall denote by

$$\mathrm{Ob}(\mathcal{C})$$

the collection of *objects* of $\mathcal{C}$. If $A \in \mathrm{Ob}(\mathcal{C})$ is an *object* of $\mathcal{C}$, then we shall denote by

$$\mathcal{C}_A$$

the category whose *objects* are morphisms $B \to A$ of $\mathcal{C}$ and whose morphisms (from an object $B_1 \to A$ to an object $B_2 \to A$) are A-morphisms $B_1 \to B_2$ in $\mathcal{C}$. Thus, we have a *natural functor*

$$(j_A)_! : \mathcal{C}_A \to \mathcal{C}$$

(given by forgetting the structure morphism to A).

We shall call an object $A \in \mathrm{Ob}(\mathcal{C})$ *terminal* if for every object $B \in \mathrm{Ob}(\mathcal{C})$, there exists a unique arrow $B \to A$ in $\mathcal{C}$.

We shall refer to a *natural transformation* between functors all of whose component morphisms are *isomorphisms* as an *isomorphism between the functors* in question. A functor $\phi : \mathcal{C}_1 \to \mathcal{C}_2$ between categories $\mathcal{C}_1$, $\mathcal{C}_2$ will be called *rigid* if ϕ has no nontrivial automorphisms. A category $\mathcal{C}$ will be called *slim* if the natural functor $\mathcal{C}_A \to \mathcal{C}$ is *rigid*, for every $A \in \mathrm{Ob}(\mathcal{C})$.

A diagram of functors between categories will be called 1-*commutative* if the various composite functors in question are *isomorphic*. When such a diagram "commutes in the literal sense" we shall say that it 0-*commutes*. Note that when a diagram in which the various composite functors are all *rigid* "1-commutes", it follows from the *rigidity* hypothesis that any isomorphism between the composite functors in question is necessarily *unique*. Thus, to state that such a diagram 1-commutes does not result in any "loss of information" by comparison to the datum of a *specific isomorphism* between the various composites in question.

We shall say that a nonempty [i.e., non-initial] object $A \in \mathrm{Ob}(\mathcal{C})$ is *connected* if it is not isomorphic to the coproduct of two nonempty objects of $\mathcal{C}$. We shall say that an object $A \in \mathrm{Ob}(\mathcal{C})$ is *mobile* (respectively, *infinitely mobile*) if there exists an object $B \in \mathrm{Ob}(\mathcal{C})$ such that the set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ has *cardinality* ≥ 2 [i.e., the diagonal from this set to the product of this set with itself is not bijective] (respectively, *infinite cardinality*). We shall say that an object $A \in \mathrm{Ob}(\mathcal{C})$ is *quasi-connected* if it is either *immobile* [i.e., not mobile] or *connected*. Thus, connected objects are always quasi-connected. If *every* object of a category $\mathcal{C}$ is *quasi-connected*, then we shall say that $\mathcal{C}$ is a *category of quasi-connected objects*. We shall say that a category $\mathcal{C}$ is *totally* (respectively, *almost totally*) *epimorphic* if every morphism in $\mathcal{C}$ whose domain is *arbitrary* (respectively, *nonempty*) and whose codomain is *quasi-connected* is an *epimorphism*.

We shall say that $\mathcal{C}$ is *of finitely (respectively, countably) connected type* if it is closed under formation of finite (respectively, countable) coproducts; every object of $\mathcal{C}$ is a coproduct of a finite (respectively, countable) collection of connected objects; and, moreover, all finite (respectively, countable) coproducts $\coprod A_i$ in the category satisfy the condition that the natural map

$$\coprod \mathrm{Hom}_{\mathcal{C}}(B, A_i) \to \mathrm{Hom}_{\mathcal{C}}(B, \coprod A_i)$$

is *bijective*, for all connected $B \in \mathrm{Ob}(\mathcal{C})$. If $\mathcal{C}$ is *of finitely or countably connected type*, then every nonempty object of $\mathcal{C}$ is *mobile*; in particular, a nonempty object of $\mathcal{C}$ is connected *if and only if* it is quasi-connected.

If a *mobile* object $A \in \mathrm{Ob}(\mathcal{C})$ satisfies the condition that every morphism in $\mathcal{C}$ whose domain is nonempty and whose codomain is A is an *epimorphism*, then A is *connected*. [Indeed, $C_1 \coprod C_2 \xrightarrow{\sim} A$, where C_1, C_2 are nonempty, implies that the composite map

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(A, B) &\hookrightarrow \mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(C_1, B) \times \mathrm{Hom}_{\mathcal{C}}(C_2, B) \\ &= \mathrm{Hom}_{\mathcal{C}}(C_1 \coprod C_2, B) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(A, B)\end{aligned}$$

is *bijective*, for all $B \in \mathrm{Ob}(\mathcal{C})$.]

If $\mathcal{C}$ is a *category of finitely or countably connected type*, then we shall write

$$\mathcal{C}^0 \subseteq \mathcal{C}$$

for the *full subcategory* of connected objects. [Note, however, that in general, objects of $\mathcal{C}^0$ are *not necessarily connected* — or even *quasi-connected* — as objects of $\mathcal{C}^0$!] On the other hand, if, in addition, $\mathcal{C}$ is *almost totally epimorphic*, then $\mathcal{C}^0$ is *totally epimorphic*, and, moreover, an object of $\mathcal{C}^0$ is *connected* [as an object of $\mathcal{C}^0$!] if and only if [cf. the argument of the preceding paragraph!] it is *mobile* [as an object of $\mathcal{C}^0$]; in particular, [assuming still that $\mathcal{C}$ is almost totally epimorphic!] every object of $\mathcal{C}^0$ is *quasi-connected* [as an object of $\mathcal{C}^0$].

If $\mathcal{C}$ is a *category*, then we shall write

$$\mathcal{C}^{\perp} \text{ (respectively, } \mathcal{C}^{\top}\text{)}$$

for the category formed from $\mathcal{C}$ by taking *arbitrary "formal" [possibly empty] finite (respectively, countable) coproducts* of objects in $\mathcal{C}$. That is to say, we define the "Hom" of $\mathcal{C}^{\perp}$ (respectively, $\mathcal{C}^{\top}$) by the formula

$$\mathrm{Hom}(\coprod_i A_i, \coprod_j B_j) \stackrel{\mathrm{def}}{=} \prod_i \coprod_j \mathrm{Hom}_{\mathcal{C}}(A_i, B_j)$$

[where the A_i, B_j are objects of $\mathcal{C}$]. Thus, $\mathcal{C}^{\perp}$ (respectively, $\mathcal{C}^{\top}$) is a *category of finitely connected type* (respectively, *category of countably connected type*). Note that objects of $\mathcal{C}$ define *connected* objects of $\mathcal{C}^{\perp}$ or $\mathcal{C}^{\top}$. Moreover, there are natural [up to isomorphism] *equivalences of categories*

$$(\mathcal{C}^{\perp})^0 \xrightarrow{\sim} \mathcal{C}; \quad (\mathcal{C}^{\top})^0 \xrightarrow{\sim} \mathcal{C}; \quad (\mathcal{D}^0)^{\perp} \xrightarrow{\sim} \mathcal{D}; \quad (\mathcal{E}^0)^{\top} \xrightarrow{\sim} \mathcal{E}$$

if $\mathcal{D}$ (respectively, $\mathcal{E}$) is a *category of finitely connected type* (respectively, *category of countably connected type*). If $\mathcal{C}$ is a *totally epimorphic category of quasi-connected objects*, then $\mathcal{C}^{\perp}$ (respectively, $\mathcal{C}^{\top}$) is an *almost totally epimorphic category of finitely (respectively, countably) connected type.*

In particular, the operations "0", "$\perp$" (respectively, "$\top$") define *one-to-one correspondences* [up to equivalence] between the *totally epimorphic categories of quasi-connected objects* and the *almost totally epimorphic categories of finitely (respectively, countably) connected type.*

Section 1: Reconstruction via the Upper Half-Plane Uniformization

In this Section, we show that the *conformal structure* of a hyperbolic Riemann surface may be functorially reconstructed — by applying the well-known geometry of the *upper half-plane uniformization* of the Riemann surface — from a certain *category of localizations* naturally associated to the Riemann surface. These categories of localizations are intended to be reminiscent of — i.e., a sort of *archimedean* analogue of — the categories of localizations of [Mzk11], §4.

In the following discussion, we shall denote the [Riemann surface constituted by the] *upper half-plane* by the notation $\mathfrak{H}$. Next, we introduce some terminology:

Definition 1.1.

(i) We shall refer to a smooth Hausdorff complex analytic stack which admits an open dense subset isomorphic to a complex manifold and [for simplicity] whose universal covering is a complex manifold as a *complex orbifold*.

(ii) We shall refer to a one-dimensional complex orbifold with at most countably many connected components as a *Riemann orbisurface*. We shall refer to a Riemann orbisurface which is a complex manifold [i.e., whose "orbifold structure" is trivial] as a *Riemann surface*.

(iii) We shall refer to a Riemann orbisurface as being *of finite type* (respectively, *of almost finite type*) if it may be obtained as the complement of a finite subset (respectively, [possibly infinite] discrete subset) in a compact Riemann orbisurface (respectively, a Riemann orbisurface of finite type).

(iv) We shall refer to a connected Riemann orbisurface X (respectively, arbitrary Riemann orbisurface X) as being an $\mathfrak{H}$*-domain* if there exists a finite [i.e., proper], surjective étale covering $X' \to X$ such that X' admits an *étale* [i.e., with derivative everywhere nonzero] holomorphic map $X' \to \mathfrak{H}$ (respectively, if every connected component of X is an $\mathfrak{H}$-domain).

(v) We shall refer to as an *RC-orbifold* [i.e., "real complex orbifold"] a pair $X^* = (X, \iota_X)$, where X is a *complex orbifold*, and ι_X is an *anti-holomorphic involution* [i.e., automorphism of order 2]; we shall refer to X as the *complexification* of the RC-orbifold X^* [cf. Remark 1.3.1 below]. Moreover, we shall append the prefix *"RC-"* to the beginning of any of the terms introduced in (i) – (iv) to refer to RC-orbifolds $X^* = (X, \iota_X)$ for which X satisfies the conditions of the term in question.

(vi) An *RC-holomorphic map*

$$X \to Y$$

between complex orbifolds X, Y is a map which is *either* holomorphic *or* anti-holomorphic at each point of X.

(vii) A *morphism between RC-orbifolds*

$$X^* = (X, \iota_X) \to Y^* = (Y, \iota_Y)$$

— where X^* is *connected* [i.e., ι_X acts *transitively* on the set of connected components of X] — is an *equivalence class of RC-holomorphic maps* $X \to Y$ compatible with ι_X, ι_Y, where we consider two RC-holomorphic maps *equivalent* if they differ by composition with ι_X [or, equivalently, ι_Y]. A *morphism between RC-orbifolds*

$$X^* = (X, \iota_X) \to Y^* = (Y, \iota_Y)$$

— where X^* is *not necessarily connected* — is the datum of a morphism of RC-orbifolds from each connected component of X^* to Y^*.

Remark 1.1.1. Note that a *Riemann orbisurface of finite type* admits a *unique algebraic structure* over $\mathbb{C}$. We refer to Lemma 1.3, (iii), for the "RC" analogue of this statement.

Remark 1.1.2. If X is an $\mathfrak{H}$-*domain*, and $Y \to X$ is an *étale morphism of complex orbifolds*, then it is immediate from the definitions that Y is also an $\mathfrak{H}$-*domain*.

Remark 1.1.3. If $Y \to X$ is a finite étale covering of connected Riemann orbisurfaces, then the *"symmetric functions"* in the various *conjugates* [i.e., with respect to the finite covering $Y \to X$] of any bounded holomorphic function on Y [e.g., a function arising from a morphism $Y \to \mathfrak{H}$] give rise to various bounded holomorphic functions on X which *determine*, up to a *finite* indeterminacy, the original bounded holomorphic function on Y.

Remark 1.1.4. For any *morphism of RC-orbifolds*

$$\Phi : X^* = (X, \iota_X) \to Y^* = (Y, \iota_Y)$$

there exists a *unique holomorphic map* $\phi : X \to Y$ lying in the equivalence class that constitutes Φ. Indeed, we may assume without loss of generality that X^* is *connected*. Then if $\phi_1 : X \to Y$ is *any* RC-holomorphic map lying in Φ, then [since X^* — but not necessarily X! — is *connected*] ϕ_1 is *either* holomorphic or anti-holomorphic. If ϕ_1 is holomorphic (respectively, anti-holomorphic), then we take $\phi \stackrel{\text{def}}{=} \phi_1$ (respectively, $\phi \stackrel{\text{def}}{=} \iota_Y \circ \phi_1 = \phi_1 \circ \iota_X$).

Proposition 1.2. (Complex Orbifolds as RC-Orbifolds)

(i) Let X be a **complex orbifold***; write X^c for its* **complex conjugate** *[i.e., holomorphic functions on X^c are anti-holomorphic functions on X]. Then*

$$\mathfrak{R} : X \mapsto (X \bigcup X^c, \iota_{\mathfrak{R}(X)})$$

— where $\iota_{\mathfrak{R}(X)}$ switches X, X^c via the [anti-holomorphic!] identification of their underlying real analytic stacks — determines a **fully faithful functor** $\mathfrak{R}$ *from the* **category of complex orbifolds and RC-holomorphic maps** *into the* **category of RC-orbifolds** *[and morphisms of RC-orbifolds].*

(ii) Let $X^* = (X, \iota_X)$ be an **RC-orbifold**. *Then there is a* **natural morphism of RC-orbifolds**

$$\mathfrak{R}(X) \to X^*$$

— which is **finite étale** *of degree 2 — given by mapping $X \subseteq X \bigcup X^c$ (respectively, $X^c \subseteq X \bigcup X^c$) to X via the identity map (respectively, ι_X).*

Proof. Immediate from the definitions. $\square$

Lemma 1.3. (Removable Singularities)

(i) No $\mathfrak{H}$**-domain** *is a Riemann orbisurface of* **almost finite type**.

(ii) A connected $\mathfrak{H}$**-domain** *is necessarily* **hyperbolic** *[i.e., its universal covering is biholomorphic to $\mathfrak{H}$].*

(iii) Any finite étale RC-holomorphic map $X \to Y$ between Riemann orbisurfaces X, Y of finite type [each of which, by Remark 1.1.1, admits a unique **algebraic structure** *over $\mathbb{C}$] is necessarily* **algebraic** *over $\mathbb{R}$. In particular, every RC-Riemann orbisurface of finite type admits a unique* **algebraic structure** *over $\mathbb{R}$.*

Proof. Assertion (i) follows immediately [cf. Remark 1.1.3] from the observation that every bounded holomorphic function on a Riemann orbisurface of almost finite type extends to a bounded holomorphic function on a Riemann orbisurface of finite type, hence to a bounded holomorphic function on a compact Riemann orbisurface, which is necessarily *constant*. Assertion (ii) follows from the same fact, applied to the case where the Riemann orbisurface of finite type in question is the *complex plane*. Assertion (iii) follows by observing that the *properness* [i.e., finiteness] assumption implies that this map $X \to Y$ extends to the *one-point compactifications* of X, Y — which possess a natural structure of [the stack-theoretic version of] *complex analytic space* [i.e., the point at infinity may be *singular*!] — and then applying the well-known fact that holomorphic [hence also RC-holomorphic] maps between algebrizable compact complex analytic spaces are necessarily *algebrizable*. $\square$

Remark 1.3.1. Thus, just as complex manifolds are an "analytic analogue" of smooth schemes over $\mathbb{C}$, RC-manifolds [i.e., "RC-orbifolds" whose stack structure is trivial] are intended to be an *analytic analogue* of *smooth schemes over* $\mathbb{R}$. Relative to this analogy, the functor $\mathfrak{R}$ of Proposition 1.2, (i), is the analogue of the functor

$$(X_{\mathbb{C}} \to \mathbb{C}) \mapsto (X_{\mathbb{C}} \to \mathbb{R})$$

that maps a smooth scheme $X_{\mathbb{C}}$ over $\mathbb{C}$ to the underlying $\mathbb{R}$-scheme. Similarly, the first datum "X" of an *RC*-complex manifold $X^* = (X, \iota_X)$, is the analogue, for a smooth scheme $X_{\mathbb{R}}$ over $\mathbb{R}$, of the associated smooth $\mathbb{C}$-scheme $X_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, and the étale double cover of Proposition 1.2, (ii), is the analogue of the étale double cover of smooth $\mathbb{R}$-schemes

$$X_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \to X_{\mathbb{R}}$$

(given by projection to the first factor).

Remark 1.3.2. Note that it follows immediately from Lemma 1.3, (iii), that every Riemann orbisurface of finite type X admits a *canonical compactification* by a compact Riemann orbisurface $\overline{X} \supseteq X$ whose "stack structure" is trivial near $\overline{X} \backslash X$. A similar statement holds for RC-Riemann orbisurfaces.

Definition 1.4. Let $X^* = (X, \iota_X)$ be an RC-orbifold. Then:

(i) We shall refer to the set $X^*(\mathbb{C})$ of points of X [i.e., points of the "coarse complex analytic space" associated to the stack X] as the set of *complex points* of X^*.

(ii) We shall refer to the set $X^*(\mathbb{R}) \subseteq X^*(\mathbb{C})$ of complex points fixed by ι_X as the set of *real points* of X^*.

(iii) We shall refer to the set $X^*[\mathbb{C}] \stackrel{\mathrm{def}}{=} X^*(\mathbb{C})/\iota_X$ of ι_X-orbits of complex points of X^* as the set of *RC-points* of X^*.

(iv) We shall refer to $\mathfrak{H}^* \stackrel{\mathrm{def}}{=} \mathfrak{R}(\mathfrak{H})$ as the *RC-upper half-plane*. We shall refer to an "RC-$\mathfrak{H}$-domain" [i.e., the "RC" version of an $\mathfrak{H}$-domain] as an *$\mathfrak{H}^*$-domain*.

Remark 1.4.1. If $X^* = (X, \iota_X)$ is a connected RC-orbifold, then one verifies easily that $X^*(\mathbb{R})$ admits a natural structure of *real analytic orbifold* whose real dimension is equal to the complex dimension of X.

Let $X^* = (X, \iota_X)$ be an *RC-orbifold*. Then note that one may consider the notion of a *covering morphism* [of RC-orbifolds] $Y^* = (Y, \iota_Y) \to (X, \iota_X)$ [i.e., $Y \to X$ is a covering morphism, in the usual sense of algebraic topology]. In particular, if X^* is *connected*, then, by considering *universal covering morphisms*, we may define the *fundamental group*

$$\pi_1(X^*)$$

of the RC-orbifold X^*.

Proposition 1.5. (Fundamental Groups of RC-orbifolds) *Let $X^* = (X, \iota_X)$ be a* **connected RC-orbifold**.

(i) If X^* arises from a **complex orbifold***, i.e., $X^* = \mathfrak{R}(X_0)$ [cf. Proposition 1.2, (i)], then we have a natural isomorphism $\pi_1(X_0) \xrightarrow{\sim} \pi_1(X^*)$. In this case, we shall say that X^* is* **of complex type**.

(ii) If X is **connected***, then we have a natural exact sequence $1 \to \pi_1(X) \to \pi_1(X^*) \to \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to 1$. Here, the surjection $\pi_1(X^*) \twoheadrightarrow \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ corresponds to the double covering of Proposition 1.2, (ii). In this case, we shall say that X^* is* **of real type**.

(iii) Suppose that X is a **hyperbolic Riemann orbisurface**. *Then $X^* \cong \mathfrak{H}^*$ if and only if $\pi_1(X^*) = \{1\}$.*

Proof. Assertions (i) and (ii), as well as the *necessity* portion of assertion (iii), are immediate from the definitions. As for the *sufficiency* portion of (iii), we observe that the condition $\pi_1(X^*) = \{1\}$ implies, by assertion (ii), that X^* arises from a *connected Riemann orbisurface* X_0. Thus, since $X = X_0 \bigcup X_0^c$ is *hyperbolic*, we conclude [from the definition of "hyperbolic"!] that $X_0 \cong \mathfrak{H}$, so $X^* \cong \mathfrak{H}^*$, as desired. $\square$

Next, let us assume that X^* is a *connected hyperbolic RC-Riemann orbisurface of finite type*. Write $\pi_1(X^*)^\wedge$ for the *profinite completion* of $\pi_1(X^*)$. Suppose that we have been given a *quotient*

$$\pi_1(X^*)^\wedge \twoheadrightarrow \Pi$$

of profinite groups. Then we may define a *category of (Π-)localizations of X^**

$$\mathfrak{Loc}_\Pi(X^*)$$

as follows: If $X^* = (X, \iota_X)$ is *of real type* (respectively, *of complex type*, and $X_0 \subseteq X$ is a connected component of X), then the *objects*

$$Y^* \text{ (respectively, } Y\text{)}$$

of this category are the RC-Riemann orbisurfaces (respectively, Riemann orbisurfaces) which are *either* $\mathfrak{H}^*$-domains (respectively, $\mathfrak{H}$-domains) *or* RC-Riemann orbisurfaces (respectively, Riemann orbisurfaces) of finite type that appear as [not necessarily connected] *finite étale coverings* of X^* (respectively, X_0) that factor through the *quotient* Π. [Here, we recall that by Lemma 1.3, (i), this "either-or" is *mutually exclusive*.] The *morphisms*

$$Y_1^* \to Y_2^* \text{ (respectively, } Y_1 \to Y_2\text{)}$$

of this category are *arbitrary étale morphisms of RC-orbifolds* (respectively, *arbitrary étale holomorphic morphisms*) which are, moreover, *proper* and *lie over X^** (respectively, X_0) whenever Y_1^*, Y_2^* (respectively, Y_1, Y_2) are *of finite type*. Thus, by Lemma 1.3, (i), [cf. also Remark 1.1.2] the codomain of any arrow with domain of finite type is also of finite type.

To keep the notation and language *simple*, even when X^* is *of complex type*, we shall regard the objects and morphisms of this category as *RC-orbifolds* and *morphisms of RC-orbifolds*, via the fully faithful functor $\mathfrak{R}$ of Proposition 1.2; moreover, thinking about things in this way renders *explicit* the *independence* of $\mathfrak{Loc}_\Pi(X^*)$ of the choice of X_0, as the notation suggests.

Lemma 1.6. **(Basic Categorical Properties)** *Let $\phi^* : Y_1^* \to Y_2^*$ be a morphism in $\mathfrak{Loc}_\Pi(X^*)$. Then:*

(i) If $\psi^* : Z_2^* \to Y_2^*$ is a morphism in $\mathfrak{Loc}_\Pi(X^*)$, then the projection morphisms

$$Y_1^* \times_{Y_2^*} Z_2^* \to Z_2^*; \quad Y_1^* \times_{Y_2^*} Z_2^* \to Y_1^*$$

obtained by forming the fibered product of Y_1^*, Z_2^* over Y_2^* in the **category of RC-orbifolds** *lie in $\mathfrak{Loc}_\Pi(X^*)$.*

(ii) ϕ^* is a **monomorphism** *if and only if it factors as the composite of an isomorphism $Y_1^* \xrightarrow{\sim} Y_3^*$ with an* **open immersion** *$Y_3^* \hookrightarrow Y_2^*$, where Y_3^* is the object determined by some open subset of $Y_2^*[\mathbb{C}]$.*

(iii) If $Y_1^* \neq \emptyset$, and Y_2^* is a **connected** *RC-orbifold, then ϕ^* is an* **epimorphism**. *In particular, the full subcategory of $\mathfrak{Loc}_\Pi(X^*)$ consisting of the connected objects is a* **totally epimorphic category of quasi-connected objects** *[cf. §0].*

Proof. Assertion (i) is immediate from the definitions if Y_1^* and Z_2^* are *of finite type*; if either Y_1^* or Z_2^* is an $\mathfrak{H}^*$*-domain*, then assertion (i) follows by applying the observation of Remark 1.1.2. Assertion (ii) may be reduced to the case where Y_2^* is *of complex type*, by base-changing [cf. assertion (i)] via the double covering of Proposition 1.2, (ii) [applied to Y_2^*]. When Y_2^* is of complex type, assertion (ii) follows immediately from the definitions, by considering various maps $\mathfrak{H}^* \to Y_2^*$. Finally, assertion (iii) follows from the elementary complex analysis fact that a holomorphic function on a connected domain which vanishes on an open subset is necessarily identically zero. □

Lemma 1.7. **(Infinitely Mobile Opens)** *Let $Y^* \in \mathrm{Ob}(\mathfrak{Loc}_\Pi(X^*))$. Write*

$$\mathfrak{Loc}_\Pi(X^*)^{\hookrightarrow}_{Y^*} \subseteq \mathfrak{Loc}_\Pi(X^*)_{Y^*}$$

for the **full subcategory** *determined by the objects constituted by arrows $Z^* \to Y^*$ which are* **monomorphisms**. *Then:*

(i) There is a natural **fully faithful functor**

$$\mathfrak{Loc}_\Pi(X^*)^{\hookrightarrow}_{Y^*} \hookrightarrow \mathrm{Open}(Y^*[\mathbb{C}])$$

[where "$\mathrm{Open}(-)$*" denotes the category whose objects are open subsets and whose morphisms are inclusions of the topological space in parentheses — cf. [Mzk10], §4] given by assigning to a monomorphism* $Z^* \rightarrowtail Y^*$ *the image of the induced map* $Z^*[\mathbb{C}] \to Y^*[\mathbb{C}]$*. This functor is an* **equivalence** *if and only if* Y^* *is an* $\mathfrak{H}^*$**-domain**.

(ii) If Y^* *is* **infinitely mobile** *[cf. §0] as an object of* $\mathfrak{Loc}_\Pi(X^*)$*, then* Y^* *is an* $\mathfrak{H}^*$**-domain**.

Proof. First, let us observe the easily verified — e.g., by *cardinality considerations* concerning the set of isomorphism classes of objects of $\mathfrak{Loc}_\Pi(X^*)$ which are *of finite type* — fact that, if Y^* is *of finite type*, then there exist *open subsets* $U \subseteq Y^*[\mathbb{C}]$ of the form $Y^*[\mathbb{C}]\backslash E$, where $E \subseteq Y^*[\mathbb{C}]$ is a *finite set*, which *do not lie in the essential image* of the functor of assertion (i) [cf. Lemma 1.3, (i)]. In light of this *observation*, assertion (i) is a formal consequence of Lemma 1.6, (ii); Remark 1.1.2. Finally, assertion (ii) is an immediate consequence of the definition of the category $\mathfrak{Loc}_\Pi(X^*)$. $\square$

Lemma 1.8. (Category-Theoreticity of the Topological Space of RC-Points) *For* $i = 1, 2$*, let* X_i^* *be a* **connected hyperbolic RC-Riemann orbisurface of finite type***;* $\pi_1(X_i^*)^\wedge \twoheadrightarrow \Pi_i$ *a* **quotient***. Let*

$$\Phi : \mathfrak{Loc}_{\Pi_1}(X_1^*) \xrightarrow{\sim} \mathfrak{Loc}_{\Pi_2}(X_2^*)$$

be an **equivalence of categories***;* $Y_i^* \in \mathrm{Ob}(\mathfrak{Loc}_{\Pi_i}(X_i^*))$*; assume that* $Y_2^* = \Phi(Y_1^*)$*. Then* Φ *induces a* **homeomorphism**

$$Y_1^*[\mathbb{C}] \xrightarrow{\sim} Y_2^*[\mathbb{C}]$$

on the topological spaces of RC-points which is **functorial** *in both* Φ *and the* Y_i^**. In particular,* Y_1^* *is* **of finite type** *if and only if* $Y_2^* = \Phi(Y_1^*)$ *is of finite type.*

Proof. Note that the *infinitely mobile objects* are manifestly *preserved* by Φ and that $\mathfrak{H}^*$ is infinitely mobile. In particular, every object of $\mathfrak{Loc}_{\Pi_i}(X_i^*)$ is *covered* by infinitely mobile opens. Thus, by functoriality [and an evident "gluing argument"], we may assume, without loss of generality, that the Y_i^* are *infinitely mobile*. But then, since the topological spaces $Y_i^*[\mathbb{C}]$ are clearly *sober*, the existence of a functorial homeomorphism as desired [as well as the fact that Φ preserves objects of finite type] follows from Lemma 1.7, (i), (ii), together with a well-known result from "topos theory" [i.e., to the effect that a sober topological space may be recovered from the category of sheaves on the space — cf., e.g., [Mzk2], Theorem 1.4]. $\square$

Lemma 1.9. (Category-Theoreticity of the Fundamental Group) *For* $i = 1, 2$*, let* X_i^**,* Π_i*,* Φ*,* Y_i^* *be as in Lemma 1.8. Then* Φ *preserves the arrows which* *are* **covering morphisms**. *In particular,* Φ *preserves isomorphs of* $\mathfrak{H}^*$ *and, if the* Y_i^* *are connected, induces an* **isomorphism of groups**

$$\pi_1(Y_1^*) \xrightarrow{\sim} \pi_1(Y_2^*)$$

— *well-defined up to composition with an* **inner automorphism** — *which is* **functorial** *in both* Φ *and the choices of universal covering morphism* $Z_i^* \to Y_i^*$ *used to define the* π_1*'s.*

Proof. Indeed, covering morphisms may be characterized by the existence of local base-changes over which the given morphism *splits* as a *disjoint union* of isomorphs of the base. Thus, the fact that Φ preserves covering morphisms follows from Lemmas 1.6, (i); 1.8. The assertion concerning *fundamental groups* then follows formally; the assertion concerning isomorphs of $\mathfrak{H}^*$ follows from Proposition 1.5, (iii). $\square$

Lemma 1.10. (Category-Theoreticity of the RC-Orbifold Structure) *For* $i = 1, 2$, *let* X_i^*, Π_i, Φ, Y_i^* *be as in Lemma 1.8. Then* Φ *induces an* **isomorphism of RC-orbifolds**

$$Y_1^* \xrightarrow{\sim} Y_2^*$$

which is **functorial** *in both* Φ *and the* Y_i^* *and* **compatible** *with the homeomorphisms of Lemma 1.8. In particular,* X_1^* *(respectively,* Y_1^**) is* **of real type** *if and only if* X_2^* *(respectively,* Y_2^**) is.*

Proof. Indeed, by *functoriality*, we may assume, without loss of generality, that the Y_i^* are *connected*. Choose *universal coverings* $Z_i^* \to Y_i^*$ [so $Z_i^* \cong \mathfrak{H}^*$] which are *compatible with* Φ [cf. Lemma 1.9]. Note that we have an *exact sequence of topological groups*

$$1 \to SL_2(\mathbb{R})/\{\pm 1\} \to \mathrm{Aut}_{\text{RC-orbifolds}}(\mathfrak{H}^*) \to \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to 1$$

— where the topology on $\mathrm{Aut}_{\text{RC-orbifolds}}(\mathfrak{H}^*)$ is that *induced* by the action of $\mathrm{Aut}_{\text{RC-orbifolds}}(\mathfrak{H}^*)$ on $\mathfrak{H}^*[\mathbb{C}]$. In particular, $\mathrm{Aut}(Z_i^*) \stackrel{\text{def}}{=} \mathrm{Aut}_{\mathfrak{Loc}_{\Pi_i}(X_i^*)}(Z_i^*)$ is *connected* if and only if X_i^* is *of complex type*. Moreover, by Lemmas 1.8, 1.9, Φ induces a *commutative diagram*

$$\begin{array}{ccc} \pi_1(Y_1^*) & \hookrightarrow & \mathrm{Aut}(Z_1^*) \\ \downarrow & & \downarrow \\ \pi_1(Y_2^*) & \hookrightarrow & \mathrm{Aut}(Z_2^*) \end{array}$$

in which the *vertical arrows* are *isomorphisms of topological groups*. Note that since $\mathrm{Aut}(Z_i^*)$ is a *real analytic Lie group*, we thus conclude [by *Cartan's theorem* — cf., e.g., [Serre], Chapter V, §9, Theorem 2] that the isomorphism $\mathrm{Aut}(Z_1^*) \xrightarrow{\sim} \mathrm{Aut}(Z_2^*)$ is, in fact, an *isomorphism of real analytic Lie groups*.

Next, let us choose *maximal connected compact subgroups* $K_i \subseteq \mathrm{Aut}(Z_i^*)$ which are compatible with Φ. Then if X_i^* is *of complex type* [so $\mathrm{Aut}(Z_i^*)$ is *connected*], then let us write $\mathrm{Aut}(Z_i^*)^0 \stackrel{\mathrm{def}}{=} \mathrm{Aut}(Z_i^*)$. On the other hand, if X_i^* is *of real type*, then we have *natural exact sequences*

$$1 \to \mathrm{Aut}(Z_i^*)^0 \to \mathrm{Aut}(Z_i^*) \to \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to 1$$

[where the superscript 0 denotes the connected component containing the identity element] which are *compatible* with Φ. Whether X_i^* is of real or complex type, let us write $K_i^0 \stackrel{\mathrm{def}}{=} K_i \bigcap \mathrm{Aut}(Z_i^*)^0$; $Y_i^* = (Y_i, \iota_{Y_i})$. Note that Y_i^* is *of real type* if and only if $\pi_1(Y_i^*) \subseteq \mathrm{Aut}(Z_i^*)$ has image $\neq \{1\}$ in $\mathrm{Aut}(Z_i^*)/\mathrm{Aut}(Z_i^*)^0$. If Y_i^* is *of real type*, then $\pi_1(Y_i) \subseteq \pi_1(Y_i^*)$ may be identified with the kernel of this map to $\mathrm{Aut}(Z_i^*)/\mathrm{Aut}(Z_i^*)^0$, and Y_i equipped with its ι_{Y_i}-action is *naturally isomorphic* to

$$K_i \backslash \mathrm{Aut}(Z_i^*)/\pi_1(Y_i)$$

[where the "/" is in the *sense of stacks*!] equipped with the natural action by $\pi_1(Y_i^*)/\pi_1(Y_i) \cong \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ [from the *right*]. If Y_i^* is *of complex type*, then Y_i^* is *naturally isomorphic* to the result of applying the functor "$\mathfrak{R}$" to the Riemann orbisurface

$$K_i^0 \backslash \mathrm{Aut}(Z_i^*)^0/\pi_1(Y_i^*)$$

[where the "/" is in the *sense of stacks*!]. Thus, we conclude that [for X_i^* of real or complex type] Φ induces an *isomorphism of RC-orbifolds* $Y_1^* \xrightarrow{\sim} Y_2^*$, as desired.

That this isomorphism is *compatible* with the homeomorphisms of Lemma 1.8 follows by *comparing the respective induced maps on "points"* — where we note that in the context of Lemma 1.8 (respectively, the present proof), *"points"* of, say, Z_i^*, amount to systems of neighborhoods of an element of $Z_i^*[\mathbb{C}]$ (respectively, left cosets of K_i in $\mathrm{Aut}(Z_i^*)$ or of K_i^0 in $\mathrm{Aut}(Z_i^*)^0$) — *by considering the action of* $\mathrm{Aut}(Z_i^*)$, K_i *on such systems of neighborhoods*. Finally, the *functoriality* of the isomorphism $Y_1^* \xrightarrow{\sim} Y_2^*$ with respect to Φ (respectively, the Y_i^*) is clear (respectively, a consequence of the compatibility with the homeomorphisms of Lemma 1.8). $\square$

Corollary 1.11. **(Preservation of Like Parity)** *For $i = 1, 2$, let X_i^*, Π_i, Φ, Y_i^* be as in Lemma 1.8; suppose further that the X_i^* are* **of real type**. *Let $Z_i^* \in \mathrm{Ob}(\mathfrak{Loc}_{\Pi_i}(X_i^*))$; assume that $Z_2^* = \Phi(Z_1^*)$, and that the Y_i^* and Z_i^* are all* **connected**. *Suppose that we are given two morphisms*

$$\phi_i, \psi_i : Z_i^* \to Y_i^*$$

in $\mathfrak{Loc}_{\Pi_i}(X_i^*)$ such that $\phi_2 = \Phi(\phi_1)$; $\psi_2 = \Phi(\psi_1)$. Then ϕ_1, ψ_1 have the **same "parity"** *— i.e., their* **unique holomorphic representatives** *[cf. Remark 1.1.4] induce the same maps on sets of connected components — if and only if ϕ_2, ψ_2 do.*

Proof. Immediate from the *functorial isomorphisms of RC-orbifolds* of Lemma 1.10. $\square$

Theorem 1.12. **(Categorical Reconstruction of Hyperbolic RC-Riemann Orbisurfaces)** *For $i = 1, 2$, let X_i^* be a* **connected hyperbolic RC-Riemann orbisurface of finite type**; *$\pi_1(X_i^*)^\wedge \twoheadrightarrow \Pi_i$ a* **quotient**. *Then the* *categories* $\mathfrak{Loc}_{\Pi_i}(X_i^*)$ *are* **slim** *[cf. §0], and, moreover, any* **equivalence of categories**

$$\Phi : \mathfrak{Loc}_{\Pi_1}(X_1^*) \xrightarrow{\sim} \mathfrak{Loc}_{\Pi_2}(X_2^*)$$

is [uniquely] **isomorphic** *[as a functor] to the equivalence induced by a* **unique isomorphism of RC-orbifolds** $X_1^* \xrightarrow{\sim} X_2^*$*. That is to say, the natural map*

$$\mathrm{Isom}^{\mathfrak{R}}((X_1^*, \Pi_1), (X_2^*, \Pi_2)) \to \mathrm{Isom}(\mathfrak{Loc}_{\Pi_1}(X_1^*), \mathfrak{Loc}_{\Pi_2}(X_2^*))$$

from isomorphisms of RC-orbifolds $X_1^* \xrightarrow{\sim} X_2^*$ *which admit [uniquely determined, up to inner automorphisms arising from* $\pi_1(X_i^*)$ *— cf. Lemma 1.9] compatible isomorphisms* $\Pi_1 \xrightarrow{\sim} \Pi_2$ *to isomorphism classes of equivalences between the categories* $\mathfrak{Loc}_{\Pi_i}(X_i^*)$ *is* **bijective***.*

Proof. Indeed, *slimness* follows, for instance, by considering the *functorial homeomorphisms* of Lemma 1.8, while the asserted *bijectivity* follows formally from the *functorial isomorphisms of RC-orbifolds* of Lemma 1.10. Here, we note that the object X_i^* of $\mathfrak{Loc}_{\Pi_i}(X_i^*)$ may be characterized, up to isomorphism, as the *object of finite type* [cf. Lemma 1.8] which forms a *terminal object* in the full subcategory of $\mathfrak{Loc}_{\Pi_i}(X_i^*)$ determined by the objects of finite type. □

Corollary 1.13. (Induced Isomorphisms of Quotients of Profinite Fundamental Groups) *In the notation of Theorem 1.12, the isomorphism*

$$\Pi_1 \xrightarrow{\sim} \Pi_2$$

induced by Φ *[well-defined up to composition with an inner automorphism of* Π_i*] is* **independent** *of the choice of* Φ*, up to the* **geometrically-induced** *automorphisms of* Π_i *— i.e., the automorphisms arising from the automorphisms of the RC-orbifold* X_i^* *that preserve the quotient* $\pi_1(X_i^*)^\wedge \twoheadrightarrow \Pi_i$*.*

Proof. A formal consequence of Theorem 1.12. □

Section 2: Categories of Parallelograms, Rectangles, and Squares

In this Section, we show that the *quasiconformal* (respectively, *conformal*; *conformal*) *structure* of a connected hyperbolic RC-Riemann orbisurface of finite type may be functorially reconstructed from a certain category of *parallelogram* (respectively, *rectangle*; *square*) *localizations*. Although, just as was the case with the categories of §1, these categories of localizations are intended to be reminiscent of the categories of localizations of [Mzk11], §4, they differ from the categories of §1 in the following *crucial* way: They admit *terminal objects* [cf. the categories of [Mzk11], §4, which also, essentially, admit terminal objects, up to finitely many automorphisms, or, alternatively, the categories called *temperoids* of [Mzk11], §3].

Definition 2.1.

(i) We shall refer to a connected hyperbolic Riemann (respectively, RC-Riemann) orbisurface as a *punctured torus* (respectively, *punctured RC-torus*) if it (respectively, each connected component of its complexification) arises as the complement of a finite, nonempty subset of a one-dimensional complex torus [i.e., the Riemann surface associated to an elliptic curve over $\mathbb{C}$]. If this finite subset is a translate of a subgroup of the complex torus (respectively, is of cardinality one), then we shall refer to the punctured torus (respectively, punctured RC-torus) as being *of torsion type* (respectively, *once-punctured*).

(ii) Let $\overline{Y}$ be a compact connected Riemann orbisurface; $Y \subseteq \overline{Y}$ the Riemann orbisurface of finite type obtained by removing some finite set S of points from $\overline{Y}$. [Thus, by Lemma 1.3, (iii), $\overline{Y}$ is completely determined by Y.] Then we shall refer to as a *logarithmic square differential* on Y a section ϕ over Y of the line bundle $\omega_Y^{\otimes 2}$ [where ω_Y is the holomorphic line bundle of differentials on Y] which extends to a section over $\overline{Y}$ of the line bundle $\omega_{\overline{Y}}^{\otimes 2}(S)$ [where $\omega_{\overline{Y}}$ is the holomorphic line bundle of differentials on $\overline{Y}$; we use the notation S to denote the reduced effective divisor on $\overline{Y}$ determined by the set S]. The *noncritical locus*

$$Y_{\text{non}} \subseteq Y$$

of a logarithmic square differential ϕ on Y is defined to be the Riemann orbisurface of points at which $\phi \neq 0$; the *universalization* of a logarithmic square differential ϕ on Y is defined to be the universal covering $\underline{Y}_{\text{non}} \to Y_{\text{non}}$ of the noncritical locus Y_{non} of ϕ. As is well-known [cf., e.g., [Lehto], Chapter IV, §6.1], if $\phi \not\equiv 0$ [i.e., ϕ is not identically zero], then the *path integral of the square root of* ϕ over $\underline{Y}_{\text{non}}$

$$\int \sqrt{\phi}$$

determines a *"natural parameter"*

$$z_\phi : \underline{Y}_{\text{non}} \to \mathbb{C}$$

on $\underline{Y}_{\text{non}}$, which is independent of the choice of square root and the choice of a basepoint for the integral, up to *multiplication by* ± 1 and *addition of a constant.* In particular, it makes sense to define a *ϕ-parallelogram* (respectively, *ϕ-rectangle*; *ϕ-square*) of $\underline{Y}_{\text{non}}$ to be an open subset or $\underline{Y}_{\text{non}}$ [or, by abuse of terminology, the associated Riemann surface] that maps bijectively via z_ϕ onto a parallelogram (respectively, rectangle; square) of $\mathbb{C}$, in the sense of Definition A.3, (i), (ii), of the Appendix. We shall refer to a ϕ-parallelogram as *pre-compact* if it is contained in a compact subset of $\underline{Y}_{\text{non}}$.

(iii) A *logarithmic square differential* ϕ^* on a connected RC-Riemann orbisurface of finite type X^* is defined to be a logarithmic square differential ϕ on [each connected component of] the complexification of X^* which is preserved by the anti-holomorphic involution of X^*. Given a logarithmic square differential ϕ^* on X^*, the noncritical locus (respectively, universalization; natural parameters [whenever $\phi \not\equiv 0$]) associated to the corresponding logarithmic square differential on the complexification of X^* thus determine a *noncritical locus* $X^*_{\text{non}} \subseteq X^*$ (respectively, *universalization* $\underline{X}^*_{\text{non}} \to X^*_{\text{non}}$; *natural parameters* $z_{\phi^*} : \underline{X}^*_{\text{non}}[\mathbb{C}] \to \mathbb{C}$) associated to ϕ^*. Here, any two natural parameters z_{ϕ^*}, z'_{ϕ^*} are related to one another as follows: z'_{ϕ^*} is equal to either $\pm z_{\phi^*} + \lambda$, for some $\lambda \in \mathbb{C}$, or the complex conjugate of this expression. In particular, we obtain a notion of *ϕ^*-parallelograms* (respectively, *ϕ^*-rectangles*; *ϕ^*-squares*; *pre-compact ϕ^*-parallelograms*) associated to ϕ^* [all of which are to be regarded as RC-Riemann surfaces over $\underline{X}^*_{\text{non}}$].

(iv) Let Y, Z be Riemann orbisurfaces of finite type. If Y, Z are connected, then we shall refer to a map $Y \to Z$ as *anti-quasiconformal* (respectively, *anti-Teichmüller*) if it is quasiconformal (respectively, a Teichmüller mapping — cf. Remark 2.1.1 below) with respect to the holomorphic structure on Y given by the holomorphic functions and the holomorphic structure on Z given by the anti-holomorphic functions. If Y, Z are not necessarily connected, then we shall refer to a map $Y \to Z$ as *RC-quasiconformal* (respectively, *RC-Teichmüller*) if its restriction to each connected component of Y determines a map to some connected component of Z that is either quasiconformal or anti-quasiconformal (respectively, either a Teichmüller mapping or an anti-Teichmüller mapping).

(v) Let $Y^* = (Y, \iota_Y)$, $Z^* = (Z, \iota_Z)$ be connected RC-Riemann orbisurfaces of finite type. Then we shall refer to as an *RC-quasiconformal morphism* (respectively, *RC-Teichmüller morphism*) $Y^* \to Z^*$ an *equivalence class* of RC-quasiconformal (respectively, RC-Teichmüller) maps $Y \to Z$ compatible with ι_Y, ι_Z, where we consider two such maps *equivalent* if they differ by composition with ι_Y [or, equivalently, ι_Z]. If Π_Y, Π_Z are *tempered topological groups*, and

$$\pi_1(Y^*) \to \Pi_Y; \quad \pi_1(Z^*) \to \Pi_Z$$

are *dense* [cf. §0] *morphisms of tempered* [cf. §0] *topological groups* [i.e., we think of $\pi_1(Y^*)$, $\pi_1(Z^*)$ as being equipped with the discrete topology, so $\pi_1(Y^*)$, $\pi_1(Z^*)$ are tempered topological groups], then we shall say that an RC-quasiconformal morphism $Y^* \to Z^*$ is *(Π_Y, Π_Z)-compatible* if there exists a [necessarily unique, by the "dense-ness" assumption] isomorphism $\Pi_Y \xrightarrow{\sim} \Pi_Z$ that is compatible [in the evident sense] with the outer isomorphism $\pi_1(Y^*) \xrightarrow{\sim} \pi_1(Z^*)$ induced by the RC-quasiconformal morphism $Y^* \to Z^*$.

(vi) A *Teichmüller pair* (X, ϕ) (respectively, *RC-Teichmüller pair* (X^*, ϕ^*)) is defined to be a pair consisting of a connected hyperbolic Riemann (respectively, RC-Riemann) orbisurface of finite type X (respectively, X^*) and a non-identically zero logarithmic square differential ϕ (respectively, ϕ^*) on X (respectively, X^*).

Remark 2.1.1. We refer to [Lehto], Chapter V, §7, §8, for more on the theory of *Teichmüller mappings* between Riemann orbisurfaces of finite type. Note that although the theory of Teichmüller mappings is typically only developed for *compact Riemann surfaces*, it extends immediately to the case of an arbitrary Riemann orbisurface of finite type Y by passing to an appropriate *Galois finite étale covering* $Z \to Y$ which extends to a ramified covering of compact Riemann orbisurfaces $\overline{Z} \to \overline{Y}$, where $\overline{Z}$ is a Riemann surface, and $\overline{Z} \to \overline{Y}$ is *ramified* at every point of $\overline{Z} \backslash Z$. [Indeed, the ramification condition implies that a logarithmic square differential on Y pulls back to a logarithmic square differential on Z which extends to a square differential *without* poles on $\overline{Z}$.]

Remark 2.1.2. Let $\Phi : Y^* \to Z^*$ be an *RC-quasiconformal morphism* (respectively, *RC-Teichmüller morphism*), as in Definition 2.1, (v) [so Y^*, Z^* are *connected*]. Then [cf. Remark 1.1.4] there exists a *unique quasiconformal map* (respectively, *Teichmüller mapping*) $\phi : Y \to Z$ lying in the equivalence class that constitutes Φ.

Remark 2.1.3. One important example of an *RC-Teichmüller pair*

$$(X^*, \phi^*)$$

is the case where X^* admits a finite étale covering $Y^* \to X^*$ such that Y^* is a *punctured RC-torus of complex type*, and the square differential $\phi^*|_{Y^*}$ extends to a square differential on the *canonical compactification* [cf. Remark 1.3.2] of Y^*. Note that in this case, ϕ^* is *completely determined*, up to a *nonzero constant multiple*. In the following, we shall refer to such a pair as *toral*. Note that if $Z^* \to X^*$ is also a finite étale covering of X^* by a punctured RC-torus of complex type Z^* such that $\phi^*|_{Z^*}$ extends to a square differential on the canonical compactification of Z^* — in which case we shall say that $Z^* \to X^*$ is *toralizing* — then one verifies immediately [by considering the *natural parameters* associated to ϕ^*] that there exists a toralizing finite étale covering $W^* \to X^*$ that *dominates* the coverings $Y^* \to X^*$, $Z^* \to X^*$. In particular, it follows that there exists a unique [up to not necessarily unique isomorphism] *"minimal toralizing finite étale covering"* $Y^*_{\mathrm{min}} \to X^*$ [i.e., such that every other toralizing finite étale covering $Y^* \to X^*$ *factors* through $Y^*_{\mathrm{min}} \to X^*$].

Let $(X^* = (X, \iota_X), \phi^*)$ be an *RC-Teichmüller pair.* Suppose that we have been given a *tempered topological group* Π and a *dense morphism*

$$\pi_1(X^*) \to \Pi$$

of tempered topological groups. Thus, for every open subgroup $H \subseteq \Pi$, the induced morphism $\pi_1(X^*) \to \Pi/H$ is *surjective*. Let us refer to a connected covering of X^* as being a Π-*covering* if it appears as a subcovering of the covering determined by such a quotient $\pi_1(X^*) \twoheadrightarrow \Pi/H$. In the following, we shall also make the following two assumptions on Π:

(1) *"Π is totally ramified at infinity"* in the sense that there exist *Galois finite* Π-*coverings* of X^* which are ramified over every point of the *canonical compactification* [cf. Remark 1.3.2] $\overline{X}^* \supseteq X^*$ which is not contained in X^*.

(2) *"Π is stack-resolving"* in the sense that there exist *Galois finite* Π-*coverings* of X^* which are of complex type and whose "stack structure" is trivial.

Now we define the *category of parallelogram (*Π*-)localizations of* (X^*, ϕ^*)

$$\mathfrak{Loc}^{\mathfrak{P}}_{\Pi}(X^*, \phi^*)$$

as follows: The *objects*

$$Z^*$$

of this category are the RC-Riemann orbisurfaces which are *either* pre-compact ϕ^*-parallelograms of the universalization $\underline{X}^*_{\mathrm{non}}$ *or* RC-Riemann orbisurfaces that appear as connected [but not necessarily finite] Π-coverings of X^*. Objects of the former type will be referred to as *parallelogram objects*; objects of the latter type will be referred to as *complete objects*. A parallelogram object defined by a ϕ^*-rectangle (respectively, ϕ^*-square) will be referred to as a *rectangle object* (respectively, *square object*). A complete object that arises from a finite covering of X^* will be referred to as a *finite object*. The *morphisms*

$$Z^*_1 \to Z^*_2$$

of this category are *arbitrary étale morphisms of RC-orbifolds over* X^* which, moreover, satisfy the property that if Z^*_1 is a *parallelogram object*, then *either* the given arrow $Z^*_1 \to Z^*_2$ is an isomorphism of RC-orbifolds *or* the given arrow $Z^*_1 \to Z^*_2$ has *pre-compact image* [i.e., the image of $Z^*_1[\mathbb{C}]$ lies inside a *compact* subset of $Z^*_2[\mathbb{C}]$].

Similarly, we define the *category of rectangle (*Π*-)localizations of* (X^*, ϕ^*)

$$\mathfrak{Loc}^{\mathfrak{R}}_{\Pi}(X^*, \phi^*)$$

to be the full subcategory of $\mathfrak{Loc}^{\mathfrak{P}}_{\Pi}(X^*, \phi^*)$ determined by the objects which are *either* complete objects *or* rectangle objects, and the *category of square (*Π*-)localizations of* (X^*, ϕ^*)

$$\mathfrak{Loc}^{\mathfrak{S}}_{\Pi}(X^*, \phi^*)$$

to be the full subcategory of $\mathfrak{Loc}^{\mathfrak{P}}_{\Pi}(X^*, \phi^*)$ determined by the objects which are *either* complete objects *or* square objects.

Observe that when X^* is *of complex type*, and we think of the objects $Z^* \to X^*$ of $\mathfrak{Loc}^{\mathfrak{P}}_{\Pi}(X^*, \phi^*)$ as being endowed with the "holomorphic structure" determined by a connected component $X_0 \subseteq X$, then all of the morphisms $Z_1^* \to Z_2^*$ of $\mathfrak{Loc}^{\mathfrak{P}}_{\Pi}(X^*, \phi^*)$ induce *holomorphic* morphisms between the connected components of the complexifications of Z_1^*, Z_2^* lying over X_0 [cf. Remark 1.1.4]. Put another way, in this case, the category $\mathfrak{Loc}^{\mathfrak{P}}_{\Pi}(X^*, \phi^*)$ may be thought of as the image via the fully faithful functor $\mathfrak{R}$ of Proposition 1.2 of a certain category of *holomorphic morphisms between Riemann orbisurfaces*. A similar statement holds for $\mathfrak{Loc}^{\mathfrak{R}}_{\Pi}(X^*, \phi^*)$, $\mathfrak{Loc}^{\mathfrak{S}}_{\Pi}(X^*, \phi^*)$.

Proposition 2.2. **(Basic Categorical Properties)** *Let $\Box$ be either "$\mathfrak{P}$", "$\mathfrak{R}$", or "$\mathfrak{S}$". Then:*

(i) The result of applying "$\top$" to the full subcategory of $\mathfrak{Loc}^{\Box}_{\Pi}(X^*, \phi^*)$ determined by the **complete** *objects is a* **connected temperoid** *[cf. [Mzk11], Definition 3.1, (ii)], with tempered fundamental group isomorphic to* Π. *In particular, it makes sense to speak of complete objects as being* **Galois** *[cf. [Mzk11], Definition 3.1, (iv)].*

(ii) The **codomain** *of any arrow of $\mathfrak{Loc}^{\Box}_{\Pi}(X^*, \phi^*)$ with* **complete domain** *is also* **complete***.*

(iii) An object Z^* of $\mathfrak{Loc}^{\Box}_{\Pi}(X^*, \phi^*)$ is **complete** *if and only if every monomorphism $Z^* \rightarrowtail W^*$ [in $\mathfrak{Loc}^{\Box}_{\Pi}(X^*, \phi^*)$] is an isomorphism.*

(iv) The object of $\mathfrak{Loc}^{\Box}_{\Pi}(X^*, \phi^*)$ determined by X^* is a **terminal object** *of the category $\mathfrak{Loc}^{\Box}_{\Pi}(X^*, \phi^*)$.*

(v) The category $\mathfrak{Loc}^{\Box}_{\Pi}(X^*, \phi^*)$ is a **totally epimorphic category of quasi-connected objects** *[cf. §0].*

(vi) The automorphism group $\mathrm{Aut}(Z^*)$ *of a complete object Z^* of $\mathfrak{Loc}^{\Box}_{\Pi}(X^*, \phi^*)$ is isomorphic to a subquotient of a group of the form Π/H, where $H \subseteq \Pi$ is an open subgroup.*

(vii) If Z^* is a **parallelogram** *object of $\mathfrak{Loc}^{\Box}_{\Pi}(X^*, \phi^*)$, then every* **endomorphism** *of Z^* [in $\mathfrak{Loc}^{\Box}_{\Pi}(X^*, \phi^*)$] is an* **automorphism***, and, moreover, the automorphism group* $\mathrm{Aut}(Z^*)$ *[of Z^* as a object of $\mathfrak{Loc}^{\Box}_{\Pi}(X^*, \phi^*)$] is* **finite***.*

(viii) Every morphism $Z_1^* \to Z_2^*$ between **parallelogram** *objects of $\mathfrak{Loc}^{\Box}_{\Pi}(X^*, \phi^*)$ is a* **monomorphism***.*

(ix) Every **monomorphism** *$Z_1^* \to Z_2^*$ of $\mathfrak{Loc}^{\Box}_{\Pi}(X^*, \phi^*)$ which is* **not** *an isomorphism factors as a composite $Z_1^* \to Z_3^* \to Z_2^*$ of* **non-isomorphisms** *$Z_1^* \to Z_3^*$, $Z_3^* \to Z_2^*$, where Z_1^*, Z_3^* are* **parallelogram** *objects.*

Proof. Assertions (i), (iv), (v), (vi) are immediate from the definitions [cf. also the proof of Lemma 1.6, (iii), in the case of assertion (v)]. To prove assertion (ii), let $Z^* \to Y^*$ be an arrow such that Z^* is complete, but Y^* is not complete. Thus, Y^* is a parallelogram object, and the morphism $Z^* \to Y^*$ is *over* X^*, hence *over* X^*_{non}. In particular, we conclude that $X^*_{\mathrm{non}} = X^*$. Note, moreover, that $Z^* \to X^*$ is a covering morphism which [outside of the category $\mathfrak{Loc}^{\square}_{\Pi}(X^*, \phi^*)$] is a subcovering of the covering $\underline{X}^*_{\mathrm{non}} \to X^*$. In particular, if we *base-change* over X^* by $\underline{X}^*_{\mathrm{non}} \to X^*$, we obtain [since Y^* is *simply connected*] a morphism $\underline{X}^*_{\mathrm{non}} \to Y^*$ *over* $\underline{X}^*_{\mathrm{non}}$, which is absurd [since, for instance, $Y^*[\mathbb{C}]$, unlike $\underline{X}^*_{\mathrm{non}}[\mathbb{C}]$, has *pre-compact* image in $\underline{X}^*_{\mathrm{non}}[\mathbb{C}]$]. In light of assertion (ii), assertion (iii) is immediate from our *pre-compactness* assumption in the definition of the morphisms of $\mathfrak{Loc}^{\square}_{\Pi}(X^*, \phi^*)$ with *parallelogram domain* [together with the observation that morphisms between complete objects are always covering morphisms, hence are monomorphisms if and only if they are isomorphisms].

Next, we consider *endomorphisms of parallelogram objects*, i.e., assertion (vii). First, let us observe that pulling back the standard volume form on $\mathbb{C}$ via a *natural parameter* yields a *volume form* $\mu_{\underline{X}_{\mathrm{non}}}$ on $\underline{X}^*_{\mathrm{non}}[\mathbb{C}]$ that is compatible with the *affine linear structure* on $\underline{X}^*_{\mathrm{non}}[\mathbb{C}]$ determined by the natural parameters, and, moreover, is held *fixed* by $\mathrm{Gal}(\underline{X}^*_{\mathrm{non}}/X^*_{\mathrm{non}})$ [since automorphisms of $\mathrm{Gal}(\underline{X}^*_{\mathrm{non}}/X^*_{\mathrm{non}})$ *fix* ϕ^*, hence map natural parameters associated to ϕ^* to natural parameters associated to ϕ^*]. In particular, since all morphisms of $\mathfrak{Loc}^{\square}_{\Pi}(X^*, \phi^*)$ are *over* X^*, it follows that $\mu_{\underline{X}_{\mathrm{non}}}$ (respectively, the affine linear structure on $\underline{X}^*_{\mathrm{non}}[\mathbb{C}]$) determines a *volume form* μ_Z (respectively, *affine linear structure*) on $Z^*[\mathbb{C}]$ that is compatible with all endomorphisms of Z^*. Thus, the fact that every endomorphism of Z^* is an automorphism follows immediately from the [easily verified, elementary] fact that every volume-preserving, affine linear automorphism of $\mathbb{C}$ that maps a parallelogram of $\underline{X}_{\mathrm{non}}$ into itself necessarily induces a bijection of this parallelogram onto itself. Moreover, it is immediate [for instance, by considering the induced bijections of edges and vertices of the closure of the parallelogram] that the group of affine linear automorphisms of this parallelogram that arise in this fashion is *finite*.

Next, we consider assertion (viii). First, observe that any two morphisms $Z^*_i \to \underline{X}^*_{\mathrm{non}}$ [where $i = 1, 2$] that arise from lifting morphisms $Z^*_i \to X^*$ of $\mathfrak{Loc}^{\square}_{\Pi}(X^*, \phi^*)$ differ by composition with an element of $\mathrm{Gal}(\underline{X}^*_{\mathrm{non}}/X^*_{\mathrm{non}})$, and that it is immediate from the definitions that there *exist* such morphisms $Z^*_i \to \underline{X}^*_{\mathrm{non}}$ which are *open immersions*. In particular, it follows that *every morphism* $Z^*_i \to \underline{X}^*_{\mathrm{non}}$ that arises from lifting a morphism $Z^*_i \to X^*$ of $\mathfrak{Loc}^{\square}_{\Pi}(X^*, \phi^*)$, hence, in particular, every composite $Z^*_1 \to Z^*_2 \to \underline{X}^*_{\mathrm{non}}$ of such a lifted morphism $Z^*_2 \to \underline{X}^*_{\mathrm{non}}$ with an arbitrary morphism $Z^*_1 \to Z^*_2$ of $\mathfrak{Loc}^{\square}_{\Pi}(X^*, \phi^*)$ is an *open immersion*. Thus, it follows immediately that any morphism $Z^*_1 \to Z^*_2$ is a monomorphism, as desired.

Finally, we consider assertion (ix). First, we recall that it is immediate from the definition of a "connected temperoid" [cf. [Mzk11], Definition 3.1, (ii)] that any monomorphism between connected objects of a connected temperoid is, in fact, an *isomorphism*. Thus, it follows from assertion (i) that Z^*_1 is a *parallelogram* object. If Z^*_2 is also a *parallelogram* object, then it follows immediately from our *pre-compactness* assumption in the definition of the morphisms of $\mathfrak{Loc}^{\square}_{\Pi}(X^*, \phi^*)$ with *parallelogram domain* that $Z^*_1 \to Z^*_2$ admits a factorization of the desired type. If, on the other hand, Z^*_2 is *complete*, then [as discussed above], the morphism

$Z_1^* \to Z_2^*$ factors as a composite $Z_1^* \to \underline{X}^*_{\text{non}} \to Z_2^*$. Now since the image of the morphism $Z_1^* \to \underline{X}^*_{\text{non}}$ is [by the definition of the "parallelogram objects" of $\mathfrak{Loc}^{\square}_{\Pi}(X^*, \phi^*)$] *pre-compact*, it follows immediately that the morphism $Z_1^* \to \underline{X}^*_{\text{non}}$ factors as a composite $Z_1^* \to Z_3^* \to \underline{X}^*_{\text{non}}$, where $Z_1^* \to Z_3^*$ is a non-isomorphism of $\mathfrak{Loc}^{\square}_{\Pi}(X^*, \phi^*)$ between parallelogram objects, and $Z_3^* \to \underline{X}^*_{\text{non}}$ is an open immersion. Thus, by composing the arrow $Z_3^* \to \underline{X}^*_{\text{non}}$ with the arrow $\underline{X}^*_{\text{non}} \to Z_2^*$, we obtain a factorization $Z_1^* \to Z_3^* \to Z_2^*$ of the desired type. This completes the proof of assertion (ix). $\square$

Theorem 2.3. **(Categorical Reconstruction of the Quasiconformal or Conformal Structure of an RC-Teichmüller pair)** *For $i = 1, 2$, let (X_i^*, ϕ_i^*) be an* **RC-Teichmüller pair***; Π_i a* **tempered topological group***;*

$$\pi_1(X_i^*) \to \Pi_i$$

a **dense** *[cf. §0]* **morphism of tempered** *[cf. §0]* **topological groups** *such that Π_i is* **"totally ramified at infinity"** *and* **"stack-resolving"** *[cf. the above discussion]. Then:*

(i) The categories $\mathfrak{Loc}^{\mathfrak{P}}_{\Pi_i}(X_i^*, \phi_i^*)$, $\mathfrak{Loc}^{\mathfrak{R}}_{\Pi_i}(X_i^*, \phi_i^*)$, $\mathfrak{Loc}^{\mathfrak{S}}_{\Pi_i}(X_i^*, \phi_i^*)$ are **slim** *[cf. §0].*

(ii) There is a natural **bijection** *between isomorphism classes of* **equivalences of categories**

$$\Phi : \mathfrak{Loc}^{\mathfrak{P}}_{\Pi_1}(X_1^*, \phi_1^*) \xrightarrow{\sim} \mathfrak{Loc}^{\mathfrak{P}}_{\Pi_2}(X_2^*, \phi_2^*)$$

and (Π_1, Π_2)**-compatible RC-Teich-müller morphisms**

$$X_1^* \xrightarrow{\sim} X_2^*$$

that "map" ϕ_1^* to a nonzero complex multiple of ϕ_2^* [i.e., ϕ_1^* (respectively, some nonzero complex multiple of ϕ_2^*) is the **"initial"** *(respectively,* **"terminal"***)* **differential** *of the RC-Teichmüller morphism — cf., e.g., [Lehto], Chapter V, Theorem 8.1]. Moreover, this bijection is obtained by considering the equivalence of categories naturally induced by such an RC-Teichmüller morphism $X_1^* \xrightarrow{\sim} X_2^*$.*

(iii) There is a natural **bijection** *between isomorphism classes of* **equivalences of categories**

$$\Phi : \mathfrak{Loc}^{\mathfrak{R}}_{\Pi_1}(X_1^*, \phi_1^*) \xrightarrow{\sim} \mathfrak{Loc}^{\mathfrak{R}}_{\Pi_2}(X_2^*, \phi_2^*)$$

and (Π_1, Π_2)**-compatible isomorphisms of RC-orbisurfaces**

$$X_1^* \xrightarrow{\sim} X_2^*$$

that map ϕ_1^* to a nonzero complex multiple of ϕ_2^*. Moreover, this bijection is obtained by considering the equivalence of categories naturally induced by such an isomorphism of RC-orbisurfaces $X_1^* \xrightarrow{\sim} X_2^*$. A similar statement holds when "$\mathfrak{Loc}^{\mathfrak{R}}$" is replaced by "$\mathfrak{Loc}^{\mathfrak{S}}$".

Proof. First, let us observe that it is immediate from the definitions that an isomorphism $X_1^* \to X_2^*$ of the type stated in assertions (ii), (iii), induces an equivalence of categories between the respective categories "$\mathfrak{Loc}^{\square}$" [where $\square$ is "$\mathfrak{P}$", "$\mathfrak{R}$", or "$\mathfrak{S}$"]. [In the case of RC-Teichmüller morphisms, this follows immediately from the manifestly affine linear *explicit local form of a Teichmüller mapping* — cf., e.g., [Lehto], Chapter V, Theorem 8.1.] In particular, we note that the definition of each of these categories is unaffected by multiplying ϕ^* by a nonzero complex number.

Next, let us suppose that we have been given an *equivalence* Φ between the respective categories "$\mathfrak{Loc}^{\square}$". Write $\mathcal{C} \stackrel{\mathrm{def}}{=} \mathfrak{Loc}^{\square}_{\Pi_i}(X_i^*, \phi_i^*)$. Let us refer to an ordered set which is isomorphic, as an ordered set, to the set of natural numbers [equipped with its usual ordering] as a *naturally ordered set*. If $W^* \in \mathrm{Ob}(\mathcal{C})$, then let us refer to as a *P-system* [i.e., a "system of parallelograms"] over W^* a projective system $\mathcal{Z} = \{Z_j^*\}_{j \in J}$

$$\ldots \to Z_{j'}^* \to \ldots \to Z_j^* \to \ldots$$

in $\mathcal{C}_{W^*}$, indexed by a naturally ordered set J, such that: (a) each object $Z_j^* \to W^*$ of this system is an arrow of $\mathcal{C}$ whose domain Z_j^* is a *parallelogram*; (b) no arrow $Z_{j'}^* \to Z_j^*$ is an isomorphism. Recall from Proposition 2.2, (viii), that every arrow $Z_{j'}^* \to Z_j^*$ is a *monomorphism*. There is an evident notion of *morphisms between P-systems over* W^* [i.e., morphisms of projective systems]. We shall call a P-system $\mathcal{Z}$ over W^* *minimal* if every morphism of P-systems [over W^*] $\mathcal{Z}' \to \mathcal{Z}$ is an isomorphism.

Let $\mathcal{Z} = \{Z_j^*\}_{j \in J}$ be a *P-system* over W^*. Then it follows from our *precompactness* assumption in the definition of the morphisms of "$\mathfrak{Loc}^{\square}$" with *parallelogram domain* that if we denote the closure of the subset

$$\boldsymbol{Z}_j \stackrel{\mathrm{def}}{=} \mathrm{Im}(Z_j^*[\mathbb{C}]) \subseteq \boldsymbol{W} \stackrel{\mathrm{def}}{=} W^*[\mathbb{C}]$$

by $\boldsymbol{K}_j \subseteq \boldsymbol{W}$, then $\boldsymbol{K}_j$ is *compact*; moreover, we have an equality

$$\boldsymbol{Z}_\infty \stackrel{\mathrm{def}}{=} \bigcap_{j \in J} \boldsymbol{Z}_j = \bigcap_{j \in J} \boldsymbol{K}_j \subseteq \boldsymbol{W}$$

of subsets of $\boldsymbol{W}$. Now suppose that for each $j \in J$, $z_j \in \boldsymbol{Z}_j$; let $z \in \boldsymbol{W}$ be a *cluster point* of the set $\{z_j\}_{j \in J}$ [i.e., some subsequence of the sequence constituted by the z_j converges to z]. [Note that since the $\boldsymbol{K}_j$ are *compact*, such a cluster point always *exists*.] Then I *claim* that $z \in \boldsymbol{Z}_\infty$. Indeed, we may assume [by replacing J by a cofinal subset of J] that $z_j \to z$. Then if we write $\boldsymbol{A}_j \stackrel{\mathrm{def}}{=} \{z_{j'}\}_{j' \geq j} \bigcup \{z\}$, then $\boldsymbol{A}_j \subseteq \boldsymbol{K}_j$, so

$$z \in \bigcap_{j \in J} \boldsymbol{A}_j \subseteq \bigcap_{j \in J} \boldsymbol{K}_j = \boldsymbol{Z}_\infty$$

as desired. In particular, since the $\boldsymbol{Z}_j$ are nonempty, it follows that $\boldsymbol{Z}_\infty$ is *nonempty*.

Now I *claim* that $\mathcal{Z} = \{Z_j^*\}_{j \in J}$ is *minimal* if and only if the cardinality $|\boldsymbol{Z}_\infty|$ of the set $\boldsymbol{Z}_\infty$ is equal to 1. Indeed, if $|\boldsymbol{Z}_\infty| > 1$, then it is immediate that one can construct a morphism of P-systems $\mathcal{Z}' \to \mathcal{Z}$ such that $\boldsymbol{Z}'_\infty \subsetneqq \boldsymbol{Z}_\infty$ [where $\boldsymbol{Z}'_\infty$ is the analogue of "$\boldsymbol{Z}_\infty$" for $\mathcal{Z}'$], so $\mathcal{Z}' \to \mathcal{Z}$ is *not* an isomorphism of P-systems. On the other hand, suppose that $|\boldsymbol{Z}_\infty| = 1$. Now since the topological space $\boldsymbol{W}$ is clearly *metrizable*, let us assume that it is equipped with a *metric* $d(-,-)$. Let

$$\mathcal{Z}' = \{Z^*_{j'}\}_{j' \in J'} \to \mathcal{Z} = \{Z^*_j\}_{j \in J}$$

be a morphism of P-systems over W^*. Thus, $\boldsymbol{Z}'_\infty = \boldsymbol{Z}_\infty$. Write $\boldsymbol{Z}_\infty = \{z\}$. Then observe that for every real $\epsilon > 0$, there exists a $j_0 \in J$ such that for all $j \geq j_0$, $\boldsymbol{Z}_j$ is contained in the set $B(z,\epsilon) \stackrel{\mathrm{def}}{=} \{w \in \boldsymbol{W} \mid d(z,w) < \epsilon\}$. Indeed, if this were false, then it would follow that for every [sufficiently large, hence every] $j \in J$, there exist $a_j, b_j \in \boldsymbol{Z}_j$ such that $d(a_j,b_j) \geq \epsilon$. Moreover, by choosing the a_j, b_j appropriately, we may assume that $a_j \to a$, $b_j \to b$, for some $a, b \in \boldsymbol{W}$. But by our discussion of cluster points in the preceding paragraph, it thus follows that $a = b = z$, hence that $\epsilon \leq d(a_j,b_j) \to d(a,b) = 0$, which is absurd. Thus, we conclude that $\boldsymbol{Z}_j \subseteq B(z,\epsilon)$ for sufficiently large $j \in J$. On the other hand, since, given a $j' \in J'$, there exists an $\epsilon > 0$ such that $B(z,\epsilon) \subseteq \boldsymbol{Z}'_{j'}$, it thus follows immediately that $\mathcal{Z}' \to \mathcal{Z}$ is an *isomorphism*, thus proving the asserted *minimality* of $\mathcal{Z}$.

Thus, in summary, we conclude that:

> There is a natural bijective correspondence between the set $W^*[\mathbb{C}]_{\mathrm{non}}$ [where the subscript "non" denotes the open subset determined by the *noncricital locus*] and the *set of isomorphism classes of minimal P-systems over* W^*.

In particular, since, by Proposition 2.2, (iii), Φ preserves parallelogram objects, we conclude that Φ induces *natural bijections*

$$W_1^*[\mathbb{C}]_{\mathrm{non}} \xrightarrow{\sim} W_2^*[\mathbb{C}]_{\mathrm{non}}$$

[where, for $i = 1, 2$, $W_i^* \in \mathrm{Ob}(\mathfrak{Loc}^{\mathfrak{P}}_{\Pi_i}(X_i^*, \phi_i^*))$, $\Phi(W_1^*) = W_2^*$] which are *functorial* in the W_i^*. Moreover, since the images of parallelograms in $W_i^*[\mathbb{C}]$ clearly form a *basis* for the topology of $W_i^*[\mathbb{C}]$, we conclude [by considering collections of isomorphism classes of P-systems over W_i^* that *factor* through some given fixed parallelogram over W_i^*] that these bijections are, in fact, *homeomorphisms*.

Note that these *functorial homeomorphisms* are already sufficient to conclude that the category $\mathcal{C}$ is *slim* [cf. the proof of slimness in Theorem 1.12 via Lemma 1.8]. This completes the proof of assertion (i).

Next, let us observe that it follows from our assumption that Π_i is *"stack-resolving"* that there *exist* finite *Galois* [cf. Proposition 2.2, (i)] W_i^* such that $\Phi(W_1^*) = W_2^*$, and, moreover, W_i^* is *of complex type*, with *trivial* "stack structure". Thus, it follows, by applying Proposition A.4 [of the Appendix] to sufficiently small pre-compact parallelogram neighborhoods of $W_i^*[\mathbb{C}]$, that, in the case of assertion (ii) (respectively, (iii)), the *functorial homeomorphism*

$$W_1^*[\mathbb{C}]_{\mathrm{non}} \xrightarrow{\sim} W_2^*[\mathbb{C}]_{\mathrm{non}}$$

constructed above is *locally affine linear* (respectively, *locally affine ortho-linear*). Now in the affine linear case, it follows from Proposition A.1 [of the Appendix], together with the *explicit local form of a Teichmüller mapping* [cf., e.g., [Lehto], Chapter V, §7 and §8, especially Theorem 8.1] that there exists an *RC-Teichmüller mapping* with domain $W_1^*[\mathbb{C}]_{\text{non}}$ and *initial differential* a nonzero complex multiple of the pull-back to $W_1^*[\mathbb{C}]_{\text{non}}$ of ϕ_1^* [which is well-defined up to possible confusion with its complex conjugate] such that the RC-holomorphic structure induced on this domain [via this RC-Teichmüller mapping] by the RC-holomorphic structure of the codomain coincides with the RC-holomorphic structure induced, via the above functorial homeomorphism, by the RC-holomorphic structure of $W_2^*[\mathbb{C}]_{\text{non}}$. In particular, this functorial homeomorphism *factors* as the composite of an *RC-Teichmüller mapping* which induces the identity map on the underlying real analytic manifolds with an *isomorphism of RC-Riemann orbisurfaces*. That is to say, the functorial homeomorphism considered above is an RC-Teichmüller mapping, as desired, hence *extends* naturally to the *canonical compactifications* [cf. Remark 1.3.2] of the $W_i^*[\mathbb{C}]_{\text{non}}$. Moreover, the functoriality of this homeomorphism [together with the fact that W_i^* is *of complex type* with *trivial* "stack structure"] allows one to *descend* the RC-Teichmüller mapping just obtained between the canonical compactifications of the $W_i^*[\mathbb{C}]_{\text{non}}$ to an RC-Teichmüller mapping between the canonical compactifications of the X_i^*, hence [by our assumption that Π_i is *totally ramified at infinity*] to an RC-Teichmüller mapping

$$X_1^* \xrightarrow{\sim} X_2^*$$

thus completing the proof of assertion (ii). [Here, we note in passing that X_i^* is *of complex type* if and only if the automorphisms of $W_i^*[\mathbb{C}]_{\text{non}}$ induced by elements of $\text{Gal}(W_i^*/X_i^*)$ preserve *some orientation* of the affine linear structure.] The affine ortho-linear case follows similarly [but is somewhat easier, since it does not involve any Teichmüller theory!]. This completes the proof of assertion (iii). $\square$

Corollary 2.4. **(The Type of a Finite Object)** *In the notation of Theorem 2.3, let*

$$\Phi : \mathfrak{Loc}^{\square}_{\Pi_1}(X_1^*, \phi_1^*) \xrightarrow{\sim} \mathfrak{Loc}^{\square}_{\Pi_2}(X_2^*, \phi_2^*)$$

[where $\square$ is "$\mathfrak{P}$", "$\mathfrak{R}$", or "$\mathfrak{S}$"] be an **equivalence of categories**. *Suppose further that, for $i = 1, 2$, W_i^* is a* **finite** *object of $\mathfrak{Loc}^{\square}_{\Pi_i}(X_i^*, \phi_i^*)$ such that $\Phi(W_1^*) = W_2^*$. Then:*

(i) W_1^* is **of complex type** *if and only if W_2^* is.*

(ii) The "stack structure" of W_1^* is **trivial** *if and only if the same is true of W_2^*.*

(iii) Suppose that the "stack structure" of W_i^* is trivial, and that W_i^* is of complex type; write $\overline{W}_i^*$ for the **canonical compactification** *[cf. Remark 1.3.2] of W_i^*. Then the* **genus** *of $\overline{W}_i^*$, as well as the* **cardinality** *of the set $\overline{W}_i^* \backslash W_i^*$ is independent of i. If, moreover, this genus is equal to 1, then the pair $(W_1^*, \phi_1^*|_{W_1^*})$*

is **toral** *[cf. Remark 2.1.3] (respectively, and* W_1^* *is* **of torsion type***) if and only if the same is true of* $(W_2^*, \phi_2^*|_{W_2^*})$ *(respectively, and* W_2^**).*

Proof. All of these assertions follow formally from Theorem 2.3, (ii), (iii). Here, we note that in the genus 1 case, *"torality"* is [easily verified to be] equivalent to the condition that the *natural parameters* [arising from the *affine linear structure*] *extend to neighborhoods of the "points at infinity"* of the canonical compactification. Once one has established "torality", the property of being *of torsion type* is completely determined by the *affine linear structure* determined by the natural parameters. □

Next, we define a somewhat different type of category of localizations, namely, a *category of finite étale localizations* [cf. the categories "Loc(−)", "$\mathrm{Loc}_k(-)$" of [Mzk7], §2]

$$\mathrm{FELoc}(X^*, \phi^*)$$

associated to an RC-Teichmüller pair (X^*, ϕ^*). The *objects* of this category are RC-Teichmüller pairs (Y^*, ψ^*), where Y^* admits a *finite étale* morphism [of RC-Riemann orbisurfaces] $Y^* \to X^*$ such that ψ^* is the pull-back to Y^* of ϕ^*. The *morphisms*

$$(Y_1^*, \psi_1^*) \to (Y_2^*, \psi_2^*)$$

are *finite étale morphisms* [of RC-Riemann orbisurfaces] $Y_1^* \to Y_2^*$ [which are *not necessarily over* X^*!] with respect to which ψ_2^* pulls back to ψ_1^*. Similarly, if X^* is *of complex type*, then one may define a similar category

$$\mathrm{FELoc}_{\mathbb{C}}(X^*, \phi^*)$$

by taking the *objects* to be the objects of $\mathrm{FELoc}(X^*, \phi^*)$ and the *morphisms* to be the "holomorphic" morphisms, i.e., the morphisms $(Y_1^*, \psi_1^*) \to (Y_2^*, \psi_2^*)$ of $\mathrm{FELoc}(X^*, \phi^*)$ that induce *holomorphic* maps from each connected component of the complexification of Y_1^* lying over some fixed connected component X_0 of the complexification of X^* to some connected component of the complexification of Y_2^* lying over X_0.

Definition 2.5. We shall refer to the RC-Teichmüller pair (X^*, ϕ^*) as a *core* (respectively, $\mathbb{C}$-*core*) if X^* is of real or complex type (respectively, of complex type), and, moreover, the object of $\mathrm{FELoc}(X^*, \phi^*)$ (respectively, $\mathrm{FELoc}_{\mathbb{C}}(X^*, \phi^*)$) determined by (X^*, ϕ^*) forms a *terminal object* of $\mathrm{FELoc}(X^*, \phi^*)$ (respectively, $\mathrm{FELoc}_{\mathbb{C}}(X^*, \phi^*)$) [cf. [Mzk7], Definition 2.1, (ii); [Mzk7], Remark 2.1.1].

Corollary 2.6. **(Extension of Equivalences of Categories)** *In the notation of Theorem 2.3, suppose further that, for* $i = 3, 4$, (X_i^*, ϕ_i^*) *is an* **RC-Teichmüller pair***, and that, for* $i = 1, 2, 3, 4$*, the morphism* $\pi_1(X_i^*) \to \Pi_i$ *is the* **identity morphism** *on* $\pi_1(X_i^*)$*. [Thus, it is immediate that* Π_i *is both "totally ramified at* *infinity" and "stack-resolving".] Moreover, for $i = 1, 2$, let us assume that we have been given an* **equivalence of categories**

$$\Phi : \mathfrak{Loc}^{\square}_{\Pi_1}(X_1^*, \phi_1^*) \xrightarrow{\sim} \mathfrak{Loc}^{\square}_{\Pi_2}(X_2^*, \phi_2^*)$$

[where $\square$ is "$\mathfrak{P}$", "$\mathfrak{R}$", or "$\mathfrak{S}$"], as well as a finite étale morphism of RC-Riemann orbisurfaces

$$(X_i^*, \phi_i^*) \to (X_{i+2}^*, \phi_{i+2}^*)$$

with respect to which ϕ_{i+2}^* pulls back to ϕ_i^*. Then:

(i) The morphism $(X_i^*, \phi_i^*) \to (X_{i+2}^*, \phi_{i+2}^*)$ induces a natural **equivalence of categories**

$$\mathfrak{Loc}^{\square}(X_i^*, \phi_i^*) \xrightarrow{\sim} \mathfrak{Loc}^{\square}(X_{i+2}^*, \phi_{i+2}^*)_{(X_i^*, \phi_i^*)}$$

[where $i = 1, 2$; $\square$ is "$\mathfrak{P}$", "$\mathfrak{R}$", or "$\mathfrak{S}$"; we omit the subscripted "Π_i's"]. In particular, we obtain a **natural functor**

$$\mathfrak{Loc}^{\square}(X_i^*, \phi_i^*) \to \mathfrak{Loc}^{\square}(X_{i+2}^*, \phi_{i+2}^*)$$

[i.e., by composing the natural functor $\mathfrak{Loc}^{\square}(X_{i+2}^*, \phi_{i+2}^*)_{(X_i^*, \phi_i^*)} \to \mathfrak{Loc}^{\square}(X_{i+2}^*, \phi_{i+2}^*)$ with the above equivalence].

(ii) Suppose that $\square$ is "$\mathfrak{R}$" or "$\mathfrak{S}$", and that, for $i = 1, 2$, $(X_{i+2}^*, \phi_{i+2}^*)$ is either a **core** *or a* **$\mathbb{C}$-core**. *Then there exists a* **1-commutative diagram**

$$\begin{array}{ccc} \mathfrak{Loc}^{\square}(X_1^*, \phi_1^*) & \to & \mathfrak{Loc}^{\square}(X_{1+2}^*, \phi_{1+2}^*) \\ \downarrow \Phi & & \downarrow \Psi \\ \mathfrak{Loc}^{\square}(X_2^*, \phi_2^*) & \to & \mathfrak{Loc}^{\square}(X_{2+2}^*, \phi_{2+2}^*) \end{array}$$

in which the vertical arrows are **equivalences of categories***; the horizontal arrows are the* **natural functors** *of (i); Ψ is* **uniquely** *determined, up to unique isomorphism, by the condition that the diagram 1-commute.*

(iii) Suppose that "$\square = \mathfrak{P}$", and that, for $i = 1, 2$, there exists a **cartesian commutative diagram** *of finite étale morphisms of RC-orbifolds*

$$\begin{array}{ccc} Y_i^* & \to & Y_{i+2}^* \\ \downarrow & & \downarrow \\ X_i^* & \to & X_{i+2}^* \end{array}$$

in which the lower horizontal arrow arises from the morphism $(X_i^*, \phi_i^*) \to (X_{i+2}^*, \phi_{i+2}^*)$ given above; Y_i^*, Y_{i+2}^* are **punctured RC-tori of complex type***; Y_{i+2}^* is* **once-punctured** *[which implies that Y_i^* is* **of torsion type**, *and that $(Y_i^*, \phi_i^*|_{Y_i^*})$,*

$(Y^*_{i+2}, \phi^*_{i+2}|_{Y^*_{i+2}})$ *are* **toral**]*; the vertical arrows are* **"minimal"** [*in the sense of Remark 2.1.3*]*. Then there exists a* **1-commutative diagram**

$$\begin{array}{ccc} \mathfrak{Loc}^{\mathfrak{P}}(X^*_1, \phi^*_1) & \rightarrow & \mathfrak{Loc}^{\mathfrak{P}}(X^*_{1+2}, \phi^*_{1+2}) \\ \downarrow \Phi & & \downarrow \Psi \\ \mathfrak{Loc}^{\mathfrak{P}}(X^*_2, \phi^*_2) & \rightarrow & \mathfrak{Loc}^{\mathfrak{P}}(X^*_{2+2}, \phi^*_{2+2}) \end{array}$$

in which the vertical arrows are **equivalences of categories***; the horizontal arrows are the* **natural functors** *of (i); Ψ is* **uniquely** *determined, up to unique isomorphism, by the condition that the diagram 1-commute.*

Proof. Assertion (i) (respectively, (ii)) is immediate from the definitions (respectively, the definitions and Theorem 2.3, (iii)). Next, we consider assertion (iii). Now it follows from Remark 2.1.3; Corollary 2.4, (i), (ii), (iii); the *minimality* assumption on $Y^*_i \rightarrow X^*_i$ [where $i = 1, 2$], that Φ maps Y^*_1 to an *isomorph* of Y^*_2. Now assertion (iii) follows by observing that the automorphism group of Y^*_i that determines the *quotient* $Y^*_i \rightarrow Y^*_{i+2}$ [hence also the quotient $Y^*_i \rightarrow X^*_{i+2}$, since the first commutative diagram in the statement of Corollary 2.6, (iii), is *cartesian*] may be recovered *category-theoretically* within $\mathfrak{Loc}^{\mathfrak{P}}(X^*_i, \phi^*_i)$, by applying Theorem 2.3, (ii) [i.e., the fact that Φ arises from a map that is compatible with the *affine linear structures* of the punctured RC-tori Y^*_i], together with the fact that Y^*_1, Y^*_2 are *of torsion type*. Finally, we note that the *uniqueness* assertions in assertions (ii), (iii) follow from the definitions, together with Theorem 2.3, (i). $\square$

Remark 2.6.1. The *"extendability"* property of Corollary 2.6, (ii), (iii), is intended to be reminiscent of the "extendability" result proven in [Mzk7], Corollary 2.5, (ii) [cf. also [Mzk8], Corollary 3.1.4, (iii); [Mzk9], Theorem 2.3; [Mzk11], Theorem 6.8, (ii)] by applying the *p-adic version of the Grothendieck Conjecture*, proven in [Mzk4].

Appendix: Quasiconformal Linear Algebra

In this Appendix, we review various well-known facts concerning the *geometry* and *linear algebra* of the euclidean plane that are relevant to the theory of quasiconformal maps.

Write

$$GL_2^{>0}(\mathbb{R}),\ GL_2^{<0}(\mathbb{R}) \subseteq GL_2(\mathbb{R})$$

for the subsets of matrices of *positive* and *negative* determinant, respectively. In the following discussion, we shall often *identify* the real vector space underlying the *complex number field* $\mathbb{C}$ with $\mathbb{R}^2$ via the bijection $\mathbb{R}^2 \ni (a,b) \mapsto a + ib \in \mathbb{C}$. This identification determines an immersion of topological groups

$$\mathbb{C}^\times \hookrightarrow GL_2^{>0}(\mathbb{R})$$

by mapping $\mathbb{C}^\times \ni a + ib \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. In the following discussion, we shall often *identify* $\mathbb{C}^\times$ with its image under this immersion and write "$\mathbb{C}^\times \subseteq GL_2^{>0}(\mathbb{R})$". The subgroup $\mathbb{C}^\times \subseteq GL_2^{>0}(\mathbb{R})$ is normalized by the matrix

$$\tau \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

conjugation by which induces *complex conjugation* on $\mathbb{C}^\times$.

If $M \in GL_2(\mathbb{R})$, then we shall write

$$f_M : \mathbb{C} \to \mathbb{C}$$

for the associated map from $\mathbb{C}$ to itself. Also, we shall often think of $GL_2(\mathbb{R})$ as acting on the *upper half-plane* $\mathfrak{H}$ in the standard fashion, via *linear fractional transformations*, i.e., if z is the standard coordinate on $\mathfrak{H}$, then $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})$ acts via the transformation

$$z \mapsto \frac{az + b}{cz + d}$$

if $M \in GL_2^{>0}(\mathbb{R})$, and via the transformation

$$z \mapsto \frac{a\overline{z} + b}{c\overline{z} + d}$$

if $M \in GL_2^{<0}(\mathbb{R})$ [cf. [Mzk5], Example 3.2].

Now we have the following:

Proposition A.1. **(The Dilatation of a Quasiconformal Map)**

(i) The map

$$t \mapsto \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$$

[where $t \in \mathbb{R}_{\geq 1} \stackrel{\mathrm{def}}{=} \{s \in \mathbb{R} \mid s \geq 1\}$] determines a structure of "one-dimensional manifold with boundary" [i.e., $\{1\}$ is the boundary of $\mathbb{R}_{\geq 1}$] on the double coset space

$$\mathbb{C}^\times \backslash GL_2^{>0}(\mathbb{R})/\mathbb{C}^\times \cong SO(2)\backslash SL_2(\mathbb{R})/SO(2)$$

— where "$\cong$" denotes the bijection induced by the natural inclusion $SL_2(\mathbb{R}) \hookrightarrow GL_2^{>0}(\mathbb{R})$.

(ii) The map

$$M \mapsto \mathrm{Dil}(M) \stackrel{\mathrm{def}}{=} \left|\frac{\partial f_M/\partial \overline{z}}{\partial f_M/\partial z}\right|$$

determines an **isomorphism of manifolds with boundary**

$$\mathbb{C}^\times \backslash GL_2^{>0}(\mathbb{R})/\mathbb{C}^\times \xrightarrow{\sim} [0,1)$$

which is given, relative to the bijection with $\mathbb{R}_{\geq 1}$ appearing in (i), by the map

$$t \mapsto \frac{t-1}{t+1}$$

[where $t \in \mathbb{R}_{\geq 1}$]. Alternatively, if we apply the bijection of $\mathfrak{H}$ with the **open unit disk** *given by $z \mapsto \frac{iz+1}{iz-1}$, then the subset $[0,1)$ of the open unit disk determines a parametrization of $\mathbb{C}^\times \backslash GL_2^{>0}(\mathbb{R})/\mathbb{C}^\times$ relative to which the map $M \mapsto \mathrm{Dil}(M)$ is given by the* **identity**.

Proof. First, we consider assertion (i). It is immediate from the definitions that the natural inclusion $SL_2(\mathbb{R}) \hookrightarrow GL_2^{>0}(\mathbb{R})$ induces a homeomorphism of coset spaces $\mathbb{C}^\times \backslash GL_2^{>0}(\mathbb{R})/\mathbb{C}^\times \cong SO(2)\backslash SL_2(\mathbb{R})/SO(2)$. Moreover, if we apply the homeomorphism $SL_2(\mathbb{R})/SO(2) \xrightarrow{\sim} \mathfrak{H}$ given by letting $SL_2(\mathbb{R})$ act on the point $i \in \mathfrak{H}$, followed by the homeomorphism discussed in assertion (ii) of $\mathfrak{H}$ with the open unit disk, then the parametrization of assertion (i) is clearly mapped onto the inverval $[0,1)$, which may be identified with the *quotient of the unit disk by the action of the unit circle* $\mathbb{S}^1 \subseteq \mathbb{C}^\times$. This completes the proof of assertion (i).

To verify assertion (ii), let us first observe that we may write $f_M(z) = c_1 z + c_2 \overline{z}$, where $c_1 = a_1 + ib_1$, $c_2 = a_2 + ib_2$; $a_1, a_2, b_1, b_2 \in \mathbb{R}$; $\mathrm{Dil}(M) = |c_2|/|c_1|$. This description of f_M, $\mathrm{Dil}(M)$ renders evident the fact that $M \mapsto \mathrm{Dil}(M)$ depends only on the image of M in $\mathbb{C}^\times \backslash GL_2^{>0}(\mathbb{R})/\mathbb{C}^\times$. Now applying $\mathrm{Dil}(-)$ to the parametrization of assertion (i) yields the function $\frac{t-1}{t+1}$ [since $2(ta+ib) = (t+1)(a+ib)+(t-1)(a-ib)$]. This completes the proof of assertion (ii). $\square$

Proposition A.2. (Dictionary between Function Theory and Linear Algebra) *Let $M \in GL_2(\mathbb{R})$. Then:*

(i) The subgroup $\mathbb{C}^\times \subseteq GL_2(\mathbb{R})$ is equal to the set of matrices $\in GL_2(\mathbb{R})$ that commute with the matrix determined by $i \in \mathbb{C}^\times$.

(ii) M lies in $\mathbb{C}^\times$ (respectively, $GL_2^{>0}(\mathbb{R})$; $\mathbb{C}^\times \cdot \tau$; $GL_2^{<0}(\mathbb{R})$) if and only if the map f_M is **conformal** *(respectively,* **quasiconformal***;* **anti-conformal***;* **anti-quasiconformal***).*

Remark A.2.1. Here, we use the term *"anti-conformal"* (respectively, *"anti-quasiconformal"*) to refer to a map that is conformal (respectively, quasiconformal) with respect to the holomorphic structure on the domain given by the holomorphic functions and the holomorphic structure on the codomain given by the anti-holomorphic functions.

Proof. Assertion (i) (respectively, (ii)) is an easy exercise (respectively, follows immediately from the definitions and assertion (i)). ☐

Definition A.3.

(i) We shall refer to any [necessarily nonempty] open subset of $\mathbb{R}^2$ given by the interior of the convex hull of the points $w, w+u, w+v, w+u+v \in \mathbb{R}^2$, where $u, v \in \mathbb{R}^2$ are a basis of $\mathbb{R}^2$ and $w \in \mathbb{R}^2$, as a *parallelogram* [of $\mathbb{R}^2$]. If $S \subseteq \mathbb{R}^2$ is a subset, then we shall refer to a parallelogram $P \subseteq S$ as *pre-compact* if it is contained in a compact subset of S [i.e., if the closure of P in $\mathbb{R}^2$ is contained in S].

(ii) We shall refer to a parallelogram of $\mathbb{R}^2$ as a *rectangle* if all of its angles are right angles [i.e., in the notation of (i), the vectors $u, v \in \mathbb{R}^2$ are *orthogonal*]. We shall refer to a rectangle of $\mathbb{R}^2$ as a *square* if all of its sides are of the same length [relative to the standard euclidean metric on $\mathbb{R}^2$].

(iii) We shall refer to a map $\mathbb{R}^2 \to \mathbb{R}^2$ as *linear* (respectively, *ortho-linear*; *quasiconformal linear*; *conformal linear*; *anti-quasiconformal linear*; *anti-conformal linear*) if it is equal to the map determined by an $M \in GL_2(\mathbb{R})$, where M is arbitrary (respectively, $\in \mathbb{C}^\times \bigcup \mathbb{C}^\times \cdot \tau$; $\in GL_2^{>0}(\mathbb{R})$; $\in \mathbb{C}^\times$; $\in GL_2^{<0}(\mathbb{R})$; $\in \mathbb{C}^\times \cdot \tau$).

(iv) We shall refer to a map $\mathbb{R}^2 \to \mathbb{R}^2$ as *affine linear* (respectively, *affine ortho-linear*; *affine quasiconformal linear*; *affine conformal linear*; *affine anti-quasiconformal linear*; *affine anti-conformal linear*) if it may be written as the composite of a *translation* [i.e., the map $\mathbb{R}^2 \to \mathbb{R}^2$ given by adding a fixed $u_0 \in \mathbb{R}^2$] with a linear (respectively, ortho-linear; quasiconformal linear; conformal linear; anti-quasiconformal linear; anti-conformal linear) map.

One way to show that some given homeomorphism of, say, $\mathbb{R}^2$ to itself is *affine linear* [i.e., either affine quasiconformal linear or affine anti-quasiconformal linear] or *affine ortho-linear* [i.e., either affine conformal linear or affine anti-conformal linear] is by applying the following result:

Proposition A.4. **(Squares, Rectangles, and Parallelograms)** *Let $B \subseteq \mathbb{R}^2$ be a* **connected open subset***; let*

$$h : B \to \mathbb{R}^2$$

be a map that determines a **homeomorphism** *of B onto a parallelogram of $\mathbb{R}^2$. Then:*

(i) Suppose that h maps pre-compact **parallelograms** *in B to parallelograms in $\mathbb{R}^2$. Then h is [the restriction to B of a map $\mathbb{R}^2 \to \mathbb{R}^2$ that is]* **affine linear***.*

(ii) Suppose that h maps pre-compact **rectangles** *in B to rectangles in $\mathbb{R}^2$. Then h is [the restriction to B of a map $\mathbb{R}^2 \to \mathbb{R}^2$ that is]* **affine ortho-linear***.*

(iii) Suppose that h maps pre-compact **squares** *in B to squares in $\mathbb{R}^2$. Then h is [the restriction to B of a map $\mathbb{R}^2 \to \mathbb{R}^2$ that is]* **affine ortho-linear***.*

Proof. First, we observe that, by considering *squares with edges parallel to the coordinate axes contained in B* and applying an appropriate affine ortho-linear map to B, we may assume without loss of generality that B itself is a *square with edges parallel to the coordinate axes centered at the origin* that contains the points (a, b), where $a, b \in \mathbb{R}$, $|a| = |b| = 1$.

Next, we consider assertion (i). Define an *"edge-segment"* of a pre-compact parallelogram $P \subseteq B$ to be an infinite set of the form $K \bigcap K'$, where K is the closure of P; K' is the closure of another pre-compact parallelogram P'; and $P \bigcap P' = \emptyset$. Consider the equivalence relation on edge-segments of P generated by the pre-equivalence relation that two edge-segments E_1, E_2 are "pre-equivalent" if the intersection $E_1 \bigcap E_2$ is *infinite*. Then observe that the *edges* of a pre-compact parallelogram $P \subseteq B$ are in natural bijective correspondence with the equivalence classes of edge-segments of P, and that, under this bijective correspondence, an edge of P is given by the *union* of edge-segments that belong to the corresponding equivalence class of edge-segments. The *vertices* of P may then be recovered as the nonempty intersections of pairs of edges. Thus, the *"affine linear structure"* of B may be recovered by considering the *combinatorics of intersections among the various edges* of the pre-compact parallelograms of B [i.e., in the notation of Definition A.3, (i), this combinatorial data encodes precisely the information that "if one takes w as the origin, then the *sum* of the points $w + u$, $w + v$ is equal to $w + u + v$"]. Since this description of the affine linear structure of B is preserved by h, we thus conclude that h is *affine linear*, as desired.

Next, we consider assertion (ii). By composing h with an appropriate affine ortho-linear map $\mathbb{R}^2 \to \mathbb{R}^2$, we may assume, without loss of generality, that h *fixes* the points $(0, 0)$ and $(1, 1)$. Next, let us observe that the ["rectangle-theoretic analogue" of the "parallelogram-theoretic"] topological description of vertices and edges given in the preceding paragraph [i.e., where "P", "P'" are assumed to be rectangles] implies that h *preserves line segments*. Since, moreover, a square may be characterized as a rectangle P such that the line segments given by the *diagonals* of the rectangle are *orthogonal* [i.e., admit sub-line segments that appear as adjacent edges of some rectangle], we conclude that h *preserves squares*. Thus, to complete the proof of assertion (ii), it suffices to verify assertion (iii).

Finally, we consider assertion (iii). By composing h with an appropriate affine ortho-linear map $\mathbb{R}^2 \to \mathbb{R}^2$, we may assume, without loss of generality, that h *fixes* the points $(0,0)$ and $(1,1)$. Since [as one verifies immediately] there is *precisely one* square $S \subseteq \mathbb{R}^2$ that has the points $(0,0)$ as $(1,1)$ as *opposite vertices*, one concludes from the topological description of vertices and edges given above that h preserves *this square* S. Thus, by possibly composing h with a *reflection* about the diagonal of S [which is manifestly an affine ortho-linear map], we may assume that h induces the *identity* morphism on the set of edges of S. Moreover, the topological description of the vertices and edges applied above also implies that h maps *line segments* in B that are *parallel to one of the two coordinate axes of* $\mathbb{R}^2$ [i.e., to one of the edges of S] to line segments in $\mathbb{R}^2$ that are parallel to one of the two coordinate axes of $\mathbb{R}^2$. On the other hand, this last property implies [in light of the fact that h induces the *identity* morphism on the set of edges of S] that h may be written in the form

$$h((a,b)) = (f(a), g(b))$$

[where f, g are real-valued continuous functions on some open interval $I \subseteq \mathbb{R}$ such that $0 \in I$ and I is preserved by multiplication by -1]. Since, moreover, h preserves squares, it follows that $f = g$.

Next, let us observe that for $a, b \in I$ such that $a, b \neq 0$, $ab \in I$, the fact that h preserves line segments [cf. the argument applied in the discussion of assertion (ii)] implies that $f(ab)/f(a)$ is *independent* of a, hence [since $f(1) = 1$] that $f(ab) = f(a) \cdot f(b)$. Since $f(0) = 0$, we thus conclude that for all $a, b \in I$ such that $ab \in I$, we have $f(ab) = f(a) \cdot f(b)$. Thus, since $\mathbb{R}^\times$ is a *real analytic Lie group*, we thus conclude [by *Cartan's theorem* — cf., e.g., [Serre], Chapter V, §9, Theorem 2] that there exists a *positive real* α such that

$$f(x) = |x|^\alpha \cdot (x/|x|)$$

for all nonzero $x \in I$. On the other hand, since, for sufficiently small $\epsilon > 0$, the function

$$x \mapsto f(x+\epsilon) - f(\epsilon)$$

satisfies *similar hypotheses* to f, we conclude that this function may be written, at least for, say, $x \in J \subseteq I$, where J is some open inverval of positive real numbers, in the form $x \mapsto c \cdot x^{\alpha'}$, for some $c, \alpha' > 0$. That is to say, we obtain the relation

$$(x+\epsilon)^\alpha - \epsilon^\alpha = c \cdot x^{\alpha'}$$

[for $x \in J$]. Thus, by, say, differentiating this relation with respect to x, taking the natural logarithm, and then differentiating again with respect to x, we obtain that

$$(\alpha - 1)x = (\alpha' - 1)(x + \epsilon)$$

[a contradiction, unless $\alpha = \alpha' = 1$]. Thus, $\alpha = \alpha' = 1$, i.e., $f(x) = x$ for all $x \in I$, so h is *affine ortho-linear*, as desired. $\square$

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